On the Non-existence of Maximal 6-arcs in Projective Planes of Order 10*

Kyliah Clarkson (karajade@uvic.ca)

University of Victoria, Victoria, B.C., Canada

A \textit{k-arc} in a projective plane of order \textit{n} is a subset of \textit{k} points such that no three of the arc points are co-incident in any line. A \textit{secant} line contains two arc points; a \textit{tangent} line contains one arc point; and an \textit{exterior} line contains no arc points. A \textit{k-arc} is \textit{complete} if every point of the plane lies on a secant.

Assume that a projective plane of order 10 exists (despite computer searches to the contrary\textsuperscript{12}), and consider a complete 6-arc in such a plane. There are 15 secants, 36 tangents, and 60 exterior lines. It can be shown that there are 15 \textit{interior} points, \textit{i.e.}, points that lie on three secants and no tangents, and 90 \textit{exterior} points, \textit{i.e.}, points that lie on one secant and four tangents. Consider any secant: it will contain two arc points, three interior points, and six exterior points. Each exterior point lies on four tangents, connecting them to the other four arc points. Each exterior point also lies on six exterior lines, which connect them, two by two, to the twelve other interior points. Ralph Denniston\textsuperscript{3} called such a set of six exterior lines through a secant’s external point a “\textit{star} on [this secant]”. Two stars are \textit{compatible} when they share no lines in common (for two stars on the same secant), or when they share no more than one line in common (for two stars on different secants). For a complete 6-arc to exist, all six stars on all fifteen secants must be mutually compatible.

Denniston found 202 non-equivalent pairs of stars to use as a starting point in a computer search, which constructed 1,043 sets of eighteen stars over three disjoint secants. Equivalence arguments reduced that to 40 sets requiring extension to a fourth secant. He found no possible extension, leading him to conclude that, if a projective plane of order 10 exists, then any 6-arc it contains must not be complete and thus must be extensible to a 7-arc.

I repeated this search, but brute-force tested all 3,136 possible starting pairs of stars in place of his equivalence arguments. My program was able to construct 5,696 sets of eighteen stars over three disjoint secants. Surprisingly, it continued on to output 192 sets of twenty-four compatible stars over four secants. When additional data was added about the possible stars of a fifth secant, my program was unable to find the required thirty compatible stars. Denniston appears to have been right, but for the wrong reason.

\begin{itemize}
\item \textsuperscript{*} M.Sc. work under supervision of Sue Whitesides (sue@uvic.ca)
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