1.3 Matrices and Matrix Operations
A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** in the matrix.
Example 1
Examples of matrices

- Some examples of matrices
  - \[
  \begin{bmatrix}
  1 & 2 \\
  3 & 0 \\
  -1 & 4
  \end{bmatrix},
  \begin{bmatrix}
  2 & 1 & 0 & -3
  \end{bmatrix},
  \begin{bmatrix}
  \ell & \pi & -\sqrt{2}
  \end{bmatrix},
  \begin{bmatrix}
  1 \\
  3
  \end{bmatrix},
  \begin{bmatrix}
  4
  \end{bmatrix}
  \]

- Size
  - \(3 \times 2\), \(1 \times 4\), \(3 \times 3\), \(2 \times 1\), \(1 \times 1\)

- Row matrix or row vector
- Column matrix or column vector

- Entries

- \# Rows
- \# Columns
A general $m \times n$ matrix $A$ as

$$A = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}$$

The entry that occurs in row $i$ and column $j$ of matrix $A$ will be denoted $a_{ij}$ or $(A)_{ij}$. If $a_{ij}$ is a real number, it is common to be referred to as **scalars**.
Matrices Notation and Terminology (2/2)

- The preceding matrix can be written as
  \[
  \begin{bmatrix}
  a_{ij} \\
  \end{bmatrix}_{m \times n}
  \text{ or }
  \begin{bmatrix}
  a_{ij}
  \end{bmatrix}
  \]

- A matrix A with n rows and n columns is called a square matrix of order n, and the shaded entries \(a_{11}, a_{22}, \cdots, a_{nn}\) are said to be on the main diagonal of A.

\[
\begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn} \\
\end{bmatrix}
\]
Two matrices are defined to be equal if they have the same size and their corresponding entries are equal.

If $A = [a_{ij}]$ and $B = [b_{ij}]$ have the same size, then $A = B$ if and only if $a_{ij} = b_{ij}$ for all $i$ and $j$. 
Example 2
Equality of Matrices

Consider the matrices

\[
A = \begin{bmatrix} 2 & 1 \\ 3 & x \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix}
\]

- If \( x = 5 \), then \( A = B \).
- For all other values of \( x \), the matrices \( A \) and \( B \) are not equal.
- There is no value of \( x \) for which \( A = C \) since \( A \) and \( C \) have different sizes.
Operations on Matrices

- If A and B are matrices of the same size, then the sum $A+B$ is the matrix obtained by adding the entries of B to the corresponding entries of A.

- Vice versa, the difference $A-B$ is the matrix obtained by subtracting the entries of B from the corresponding entries of A.

- Note: Matrices of different sizes cannot be added or subtracted.

$$
(A + B)_{ij} = (A)_{ij} + (B)_{ij} = a_{ij} + b_{ij}
$$

$$
(A - B)_{ij} = (A)_{ij} - (B)_{ij} = a_{ij} - b_{ij}
$$
Example 3
Addition and Subtraction

- Consider the matrices

\[
A = \begin{bmatrix}
2 & 1 & 0 & 3 \\
-1 & 0 & 2 & 4 \\
4 & -2 & 7 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
-4 & 3 & 5 & 1 \\
2 & 2 & 0 & -1 \\
3 & 2 & -4 & 5
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 1 \\
2 & 2
\end{bmatrix}
\]

- Then

\[
A + B = \begin{bmatrix}
-2 & 4 & 5 & 4 \\
1 & 2 & 2 & 3 \\
7 & 0 & 3 & 5
\end{bmatrix}, \quad A - B = \begin{bmatrix}
6 & -2 & -5 & 2 \\
-3 & -2 & 2 & 5 \\
1 & -4 & 11 & -5
\end{bmatrix}
\]

- The expressions A+C, B+C, A-C, and B-C are undefined.
Definition

If $A$ is any matrix and $c$ is any scalar, then the product $cA$ is the matrix obtained by multiplying each entry of the matrix $A$ by $c$. The matrix $cA$ is said to be the **scalar multiple** of $A$.

In matrix notation, if $A = \begin{bmatrix} a_{ij} \end{bmatrix}$, then

$$(cA)_{ij} = c(A)_{ij} = ca_{ij}$$
Example 4
Scalar Multiples (1/2)

For the matrices

\[
A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix}, \quad C = \begin{bmatrix} 9 & -6 & 3 \\ 3 & 0 & 12 \end{bmatrix}
\]

We have

\[
2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}, \quad (-1)B = \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix}, \quad \frac{1}{3}C = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{bmatrix}
\]

It is common practice to denote \((-1)B\) by \(-B\).
Example 4
Scalar Multiples (2/2)

If $A_1, A_2, \ldots, A_n$ are matrices of the same size and $c_1, c_2, \ldots, c_n$ are scalars, then an expression of the form

$$c_1 A_1 + c_2 A_2 + \cdots + c_n A_n$$

is called a **linear combination** of $A_1, A_2, \ldots, A_n$ with **coefficients** $c_1, c_2, \ldots, c_n$. For example, if $A$, $B$, and $C$ are the matrices in Example 4, then

$$2A - B + \frac{1}{3}C = 2A + (-1)B + \frac{1}{3}C$$

$$= \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix} + \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 2 \\ 4 & 3 & 11 \end{bmatrix}$$

is the linear combination of $A$, $B$, and $C$ with scalar coefficients 2, $-1$, and $\frac{1}{3}$.
Definition

- If $A$ is an $m \times r$ matrix and $B$ is an $r \times n$ matrix, then the **product** $AB$ is the $m \times n$ matrix whose entries are determined as follows.
- To find the entry in row $i$ and column $j$ of $AB$, single out row $i$ from the matrix $A$ and column $j$ from the matrix $B$. Multiply the corresponding entries from the row and column together and then add up the resulting products.
Example 5
Multiplying Matrices (1/2)

Consider the matrices

\[ A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} \]

Solution

Since \( A \) is a \( 2 \times 3 \) matrix and \( B \) is a \( 3 \times 4 \) matrix, the product \( AB \) is a \( 2 \times 4 \) matrix. And:

\[
AB = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1r} \\
    a_{21} & a_{22} & \cdots & a_{2r} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{i1} & a_{i2} & \cdots & a_{ir} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mr}
\end{bmatrix}\begin{bmatrix}
    b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\
    b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\
    \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn}
\end{bmatrix}
\]

(4)

the entry \((AB)_{ij}\) in row \(i\) and column \(j\) of \(AB\) is given by

\[
(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ir}b_{rj}
\]

(5)
Example 5

Multiplying Matrices (2/2)

\[
\begin{bmatrix}
1 & 2 & 4 \\
2 & 6 & 0
\end{bmatrix}
\begin{bmatrix}
4 & 1 & 4 & 3 \\
0 & -1 & 3 & 1 \\
2 & 7 & 5 & 2
\end{bmatrix} = 
\begin{bmatrix}
\hline
\hline
\hline
\hline
\hline
\end{bmatrix}
\]

\[(2 \cdot 4) + (6 \cdot 3) + (0 \cdot 5) = 26\]

The entry in row 1 and column 4 of \(AB\) is computed as follows.

\[
\begin{bmatrix}
1 & 2 & 4 \\
2 & 6 & 0
\end{bmatrix}
\begin{bmatrix}
4 & 1 & 4 & 3 \\
0 & -1 & 3 & 1 \\
2 & 7 & 5 & 2
\end{bmatrix} = 
\begin{bmatrix}
\hline
\hline
\hline
\hline
\hline
\end{bmatrix}
\]

\[(1 \cdot 3) + (2 \cdot 1) + (4 \cdot 2) = 13\]

The computations for the remaining products are

\[(1 \cdot 4) + (2 \cdot 0) + (4 \cdot 2) = 12\]

\[(1 \cdot 1) - (2 \cdot 1) + (4 \cdot 7) = 27\]

\[(1 \cdot 4) + (2 \cdot 3) + (4 \cdot 5) = 30\]

\[(2 \cdot 4) + (6 \cdot 0) + (0 \cdot 2) = 8\]

\[(2 \cdot 1) - (6 \cdot 1) + (0 \cdot 7) = -4\]

\[(2 \cdot 3) + (6 \cdot 1) + (0 \cdot 2) = 12\]

\[AB = \begin{bmatrix}
12 & 27 & 30 & 13 \\
8 & -4 & 26 & 12
\end{bmatrix}\]
Examples 6
Determining Whether a Product Is Defined

- Suppose that $A$, $B$, and $C$ are matrices with the following sizes:

  $$
  \begin{array}{ccc}
  A & B & C \\
  3 \times 4 & 4 \times 7 & 7 \times 3 \\
  \end{array}
  $$

- Solution:
  - Then by (3), $AB$ is defined and is a $3 \times 7$ matrix; $BC$ is defined and is a $4 \times 3$ matrix; and $CA$ is defined and is a $7 \times 4$ matrix. The products $AC$, $CB$, and $BA$ are all undefined.
Partitioned Matrices

A matrix can be subdivided or partitioned into smaller matrices by inserting horizontal and vertical rules between selected rows and columns.

- For example, below are three possible partitions of a general $3 \times 4$ matrix $A$.
  - The first is a partition of $A$ into four submatrices $A_{11}, A_{12}, A_{21},$ and $A_{22}$.
  - The second is a partition of $A$ into its row matrices $r_1, r_2,$ and $r_3$.
  - The third is a partition of $A$ into its column matrices $c_1, c_2, c_3,$ and $c_4$.

\[
A = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{21} & a_{22} & a_{23} & a_{24} \\
  a_{31} & a_{32} & a_{33} & a_{34}
\end{bmatrix} = \begin{bmatrix}
  A_{11} & A_{12} \\
  A_{21} & A_{22}
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{21} & a_{22} & a_{23} & a_{24} \\
  a_{31} & a_{32} & a_{33} & a_{34}
\end{bmatrix} = \begin{bmatrix}
  r_1 \\
  r_2 \\
  r_3
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{21} & a_{22} & a_{23} & a_{24} \\
  a_{31} & a_{32} & a_{33} & a_{34}
\end{bmatrix} = \begin{bmatrix}
  c_1 & c_2 & c_3 & c_4
\end{bmatrix}
\]
Matrix Multiplication by columns and by Rows

Sometimes it may be desirable to find a particular row or column of a matrix product $AB$ without computing the entire product.

If $a_1, a_2, ..., a_m$ denote the row matrices of $A$ and $b_1, b_2, ..., b_n$ denote the column matrices of $B$, then it follows from Formulas (6) and (7) that

\[
AB = A[b_1 \ b_2 \ \cdots \ b_n] = [Ab_1 \ Ab_2 \ \cdots \ Ab_n]
\]

For $AB$ computed column by column:

\[
AB = \begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_m
\end{bmatrix}
\]

For $AB$ computed row by row:

\[
B = \begin{bmatrix}
a_1 B \\
a_2 B \\
\vdots \\
a_m B
\end{bmatrix}
\]
Example 7

Example 5 Revisited

- This is the special case of a more general procedure for multiplying partitioned matrices.
- If $A$ and $B$ are the matrices in Example 5, then from (6) the second column matrix of $AB$ can be obtained by the computation:

\[
\begin{bmatrix}
1 & 2 & 4 \\
2 & 6 & 0
\end{bmatrix}
\begin{bmatrix}
1 \\
-1 \\
7
\end{bmatrix}
= \begin{bmatrix}
27 \\
-4
\end{bmatrix}
\]

- From (7) the first row matrix of $AB$ can be obtained by the computation:

\[
\begin{bmatrix}
1 & 2 & 4 \\
0 & -1 & 3 \\
2 & 7 & 5
\end{bmatrix}
\begin{bmatrix}
4 & 1 & 4 & 3 \\
0 & -1 & 3 & 1 \\
2 & 7 & 5 & 2
\end{bmatrix}
= \begin{bmatrix}
12 & 27 & 30 & 13
\end{bmatrix}
\]
Matrix Products as Linear Combinations (1/2)

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\quad \text{and} \quad
x = \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}
\]

Then

\[
Ax = \begin{bmatrix}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n
\end{bmatrix}
= x_1 \begin{bmatrix}
a_{11} \\
a_{21} \\
\vdots \\
a_{m1}
\end{bmatrix} + x_2 \begin{bmatrix}
a_{12} \\
a_{22} \\
\vdots \\
a_{m2}
\end{bmatrix} + \cdots + x_n \begin{bmatrix}
a_{1n} \\
a_{2n} \\
\vdots \\
a_{mn}
\end{bmatrix}
\]

\[\text{(10)}\]
In words, (10) tells us that the product $A \mathbf{x}$ of a matrix $A$ with a column matrix $\mathbf{x}$ is a linear combination of the column matrices of $A$ with the coefficients coming from the matrix $\mathbf{x}$.

In the exercises we ask the reader to show that the product $\mathbf{y} A$ of a $1 \times m$ matrix $\mathbf{y}$ with an $m \times n$ matrix $A$ is a linear combination of the row matrices of $A$ with scalar coefficients coming from $\mathbf{y}$.
Example 8

Linear Combination

The matrix product

\[
\begin{bmatrix}
-1 & 3 & 2 \\
1 & 2 & -3 \\
2 & 1 & -2
\end{bmatrix}
\begin{bmatrix}
2 \\
-1 \\
3
\end{bmatrix} =
\begin{bmatrix}
1 \\
-9 \\
-3
\end{bmatrix}
\]

can be written as the linear combination of column matrices

\[
2 \begin{bmatrix}
-1 \\
1 \\
2
\end{bmatrix} - 1 \begin{bmatrix}
3 \\
2 \\
1
\end{bmatrix} + 3 \begin{bmatrix}
2 \\
-3 \\
-2
\end{bmatrix} =
\begin{bmatrix}
1 \\
-9 \\
-3
\end{bmatrix}
\]

The matrix product

\[
\begin{bmatrix}
1 & -9 & -3
\end{bmatrix}
\begin{bmatrix}
-1 & 3 & 2 \\
1 & 2 & -3 \\
2 & 1 & -2
\end{bmatrix} =
\begin{bmatrix}
-16 & -18 & 35
\end{bmatrix}
\]

can be written as the linear combination of row matrices

\[
1\begin{bmatrix}
-1 & 3 & 2
\end{bmatrix} - 9\begin{bmatrix}
1 & 2 & -3
\end{bmatrix} - 3\begin{bmatrix}
2 & 1 & -2
\end{bmatrix} =
\begin{bmatrix}
-16 & -18 & 35
\end{bmatrix}
\]
Example 9
Columns of a Product $AB$ as Linear Combinations

We showed in Example 5 that

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \\ 2 & 7 & 5 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

The column matrices of $AB$ can be expressed as linear combinations of the column matrices of $A$ as follows:

$$\begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 27 \\ -4 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 7 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 30 \\ 26 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 5 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 13 \\ 12 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$
Consider any system of $m$ linear equations in $n$ unknowns.

Since two matrices are equal if and only if their corresponding entries are equal.

The $m \times 1$ matrix on the left side of this equation can be written as a product to give:
Matrix form of a Linear System (1/2)

- If \( w \) designate these matrices by \( A, \mathbf{x}, \) and \( \mathbf{b} \), respectively, the original system of \( m \) equations in \( n \) unknowns has been replaced by the single matrix equation
  \[
  A\mathbf{x} = \mathbf{b}
  \]
- The matrix \( A \) in this equation is called the coefficient matrix of the system. The augmented matrix for the system is obtained by adjoining \( \mathbf{b} \) to \( A \) as the last column; thus the augmented matrix is

\[
[A | \mathbf{b}] = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
  a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{bmatrix}
\]
Definition

- If $A$ is any $m \times n$ matrix, then the transpose of $A$, denoted by $A^T$, is defined to be the $n \times m$ matrix that results from interchanging the rows and columns of $A$; that is, the first column of $A^T$ is the first row of $A$, the second column of $A^T$ is the second row of $A$, and so forth.
Example 10
Some Transposes (1/2)

The following are some examples of matrices and their transposes.

\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34}
\end{bmatrix}, \quad
B = \begin{bmatrix}
2 & 3 \\
1 & 1 \\
5 & 6
\end{bmatrix}, \quad
C = [1 \quad 3 \quad 5], \quad D = [4]
\]

\[
A^T = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33} \\
a_{14} & a_{24} & a_{34}
\end{bmatrix}, \quad
B^T = \begin{bmatrix}
2 & 1 & 5 \\
3 & 4 & 6
\end{bmatrix}, \quad
C^T = \begin{bmatrix}
1 \\
3 \\
5
\end{bmatrix}, \quad D^T = [4]
\]
Example 10
Some Transposes (2/2)

- Observe that \((A^T)_{ij} = (A)_{ji}\)

- In the special case where \(A\) is a **square** matrix, the transpose of \(A\) can be obtained by interchanging entries that are symmetrically positioned about the main diagonal.

\[
A = \begin{bmatrix}
1 & -2 & 4 \\
3 & 7 & 0 \\
-5 & 8 & 6
\end{bmatrix}
\rightarrow \begin{bmatrix}
1 & -2 & 4 \\
3 & 7 & 0 \\
-5 & 8 & 6
\end{bmatrix}
\rightarrow A^T = \begin{bmatrix}
1 & 3 & -5 \\
-2 & 7 & 8 \\
4 & 0 & 6
\end{bmatrix}
\]
Definition

- If $A$ is a square matrix, then the trace of $A$, denoted by $\text{tr}(A)$, is defined to be the sum of the entries on the main diagonal of $A$. The trace of $A$ is undefined if $A$ is not a square matrix.
Example 11
Trace of Matrix

The following are examples of matrices and their traces.

\[ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix} \]

\[ \text{tr}(A) = a_{11} + a_{22} + a_{33} \]
\[ \text{tr}(B) = -1 + 5 + 7 + 0 = 11 \]
Reference

- vision.ee.ccu.edu.tw/modules/tinyd2/content/93_LA/Chapter1(1.1~1.3).ppt