Computational Completeness

1 Definitions and examples

Let \( \Sigma = \{ f_1, f_2, \ldots, f_i, \ldots \} \) be a (finite or infinite) set of Boolean functions. Any of the functions \( f_i \in \Sigma \) can be a function of arbitrary number of arguments.

Definition 1 The set \( \Sigma \) is called computationally complete (or, simply, complete), if any Boolean function can be expressed as a formula involving just the functions of the set \( \Sigma \).

Example 1 The set \( \Sigma_1 = \{ \overline{x}_1, x_1 \lor x_2, x_1 \land x_2 \} \) is complete, because any Boolean function can be represented in the SOP or in the POS form, and these representations involve just the functions of \( \Sigma_1 \).

Example 2 The set \( \Sigma_2 = \{ \overline{x}_1, x_1 \land x_2 \} \) is complete, because \( x_1 \lor x_2 = \overline{x}_1 \land \overline{x}_2 \). Therefore, the completeness of \( \Sigma_2 \) follows from the completeness of \( \Sigma_1 \).

Example 3 The set \( \Sigma_3 = \{ x_1|x_2 \} \), where \( x_1|x_2 = \overline{x}_1 \land x_2 \), is complete. Indeed,
\[
x_1|x_1 = \overline{x}_1, \quad (x_1|x_2)|(x_1|x_2) = x_1 \land x_2.
\]
Thus, the question concerning the completeness of \( \Sigma_3 \) is reduced to one of \( \Sigma_2 \).

Example 4 The set \( \Sigma_4 = \{ 1, x_1 \land x_2, x_1 \oplus x_2 \} \), where \( x_1 \oplus x_2 \) is the XOR function and 1 is the constant function, is complete. Indeed, \( x_1 \oplus 1 = \overline{x} \). Hence, the completeness of \( \Sigma_4 \) follows from the completeness of \( \Sigma_2 \).

Example 5 The set \( \Sigma_5 = \{ 1, x_1 \land x_2 \} \) is not complete, because any function that can be expressed by a formula involving just the functions of \( \Sigma_5 \) is either the constant function 1 or the function of the form \( x_1 \land x_2 \land \cdots \land x_n \) for \( n = 2, 3, \ldots \).

Given a set \( \Sigma \) of Boolean functions, how to recognize if \( \Sigma \) is complete? In order to present a complete answer to this question we introduce 5 following classes of Boolean functions: \( T_0, T_1 \), \( L, S, \) and \( M \).

2 The class \( T_0 \)

Definition 2 The class \( T_0 \) consists of all Boolean functions \( f \) (of any number of arguments) defined as follows:
\[
T_0 = \{ f(x_1, \ldots, x_n) \mid f(0, 0, \ldots, 0) = 0 \}.
\]
Example 6 The following functions belong to the class $T_0$: $0$, $x_1 \land x_2$, $x_1 \lor x_2$, $x_1 \oplus x_2$.

Example 7 The functions 1 and $\overline{x}_1$ are not in $T_0$.

The number of function in $T_0$ which depend on $n$ variables is $2^{2^n-1} = \frac{1}{2} \cdot 2^n$.

3 The class $T_1$

Definition 3 The class $T_1$ consists of all Boolean functions $f$ (of any number of arguments) defined as follows:

$$T_1 = \{ f(x_1, \ldots, x_n) \mid f(1, 1, \ldots, 1) = 1 \}.$$  

Example 8 The following functions belong to the class $T_1$: 1, $x_1 \land x_2$, $x_1 \lor x_2$.

Example 9 The functions $x_1 \oplus x_2$ and $\overline{x}_1$ are not in $T_1$.

The number of function in $T_1$ which depend on $n$ variables also equals $2^{2^n-1} = \frac{1}{2} \cdot 2^n$.

4 The class $L$ of linear functions

Definition 4 The class $L$ consists of functions (of any number of arguments) that can be represented in the form

$$L = \{ f(x_1, \ldots, x_n) \mid f = (a_1 \land x_1) \oplus (a_2 \land x_2) \oplus \cdots \oplus (a_n \land x_n) \oplus b \},$$

where $a_1, \ldots, a_n, b \in \{0, 1\}$ are some fixed constants.

Example 10 Since $\overline{x}_1 = x_1 \oplus 1$, then $\overline{x}_1 \in L$.

Example 11 $x_1 \lor x_2 \notin L$. Indeed, assume the contrary, i.e. $x_1 \lor x_2 \in L$. Then $x_1 \lor x_2 = (a_1 \land x_1) \oplus (a_2 \land x_2) \oplus b$ for some constants $a_1, a_2, b \in \{0, 1\}$. Since $x_1 \lor x_2$ significantly depends on two variables (i.e. cannot be represented as a function of one or less variables), then $a_1 = a_2 = 1$, because otherwise we would get a function depending on just one variable. Hence, our representation should be of the form $x_1 \lor x_2 = x_1 \oplus x_2 \oplus b$ for some $b \in \{0, 1\}$.

Now if we assume $b = 0$, then $x_1 \lor x_2 = x_1 \oplus x_2$, which is a contradiction. Otherwise, if $b = 1$, then $x_1 \lor x_2 = x_1 \oplus x_2 \oplus 1 = \overline{x}_1 \oplus x_2$, which is a contradiction too. The only way to avoid a contradiction is to accept $x_1 \lor x_2 \notin L$.

Similarly $x_1 \land x_2 \notin L$.

The number of functions in $L$ which depend of $n$ variables is $2^{n+1}$, because any such a function can be encoded by the binary vector $(a_1, a_2, \ldots, a_n, b)$ consisting of $n + 1$ entries.
5 The class $S$ of self-dual functions

**Definition 5** A Boolean function is called **self-dual** if

$$f(x_1, \ldots, x_n) = \bar{f}(\bar{x}_1, \ldots, \bar{x}_n)$$

for any $x_1, \ldots, x_n \in \{0, 1\}$.

The class $S$ consists of all self-dual Boolean functions (of any number of variables).

**Example 12** Obviously, $\bar{x}_1 \in S$. A more complicated example is the majority function

$$f(x_1, x_2, x_3) = x_1 x_2 \lor x_1 x_3 \lor x_2 x_3.$$  

Indeed, using the DeMorgan’s theorem

$$\bar{f}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \frac{x_1 x_2 \lor x_1 x_3 \lor x_2 x_3}{x_1 x_2 \lor \bar{x}_1 \bar{x}_3 \lor \bar{x}_2 \bar{x}_3} = (x_1 \lor x_2)(x_1 \lor x_3)(x_2 \lor x_3) = x_1 x_2 \lor x_1 x_3 \lor x_2 x_3 = f(x_1, x_2, x_3).$$

**Example 13** The functions $f_1(x_1, x_2) = x_1 \land x_2$ and $f_2(x_1, x_2) = x_1 \lor x_2$ do not belong to $S$. Indeed, $f_1(0, 1) = f_1(1, 0)$ and $f_2(0, 1) = f_2(1, 0)$.

What is the number of the self-dual functions depending on $n$ variables? To compute this number, represent the function $f(x_1, \ldots, x_n) \in S$ by the logical table with $2^n$ rows, which we split into two equal parts, consisting of $2^n - 1$ rows each:

<table>
<thead>
<tr>
<th>x_1 x_2 \cdots x_n</th>
<th>f(x_1, \ldots, x_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 \cdots 0 0</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>0 0 \cdots 0 1</td>
<td>$\bar{\alpha}$</td>
</tr>
<tr>
<td>\cdots</td>
<td></td>
</tr>
<tr>
<td>0 1 \cdots 1 1</td>
<td></td>
</tr>
<tr>
<td>1 0 \cdots 0 0</td>
<td></td>
</tr>
<tr>
<td>1 0 \cdots 0 1</td>
<td></td>
</tr>
<tr>
<td>\cdots</td>
<td></td>
</tr>
<tr>
<td>1 1 \cdots 1 1</td>
<td></td>
</tr>
</tbody>
</table>

Note that the $i$th row of the left part of the table is the negation of the $(2^n - i)^{th}$ row. If $f(x_1, \ldots, x_n) \in S$ then the value of $f$ in these rows are different. Therefore, $f$ is completely determined by the values it takes on just in the upper (or just lower) part of the table. In other words, the number of the self-dual functions in question equals the number of binary strings of length $2^n - 1$, i.e. $2^{2^n - 1} = \sqrt{2^{2^n}}$. 

3
6 The class \( M \) of monotone functions

**Definition 6** Let \((x_1, \ldots, x_n)\) and \((y_1, \ldots, y_n)\) be two binary vectors of the same dimension. We write \((x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)\) if \(x_i \leq y_i\) for \(i = 1, 2, \ldots, n\).

If \((x_1, \ldots, x_n) \not\leq (y_1, \ldots, y_n)\) and \((y_1, \ldots, y_n) \not\leq (x_1, \ldots, x_n)\) then we say that these vectors are incompatible.

**Example 14** It holds: \((0, 1, 0) \leq (1, 1, 0)\), and \((0, 0, \ldots, 0) \leq (1, 1, \ldots, 1)\).

**Example 15** The vectors \((0, 1, 0)\) and \((1, 0, 0)\) are incompatible. In general, a vector and its binary coordinatewise negation are incompatible, cf. e.g. \((0, 1, 0)\) and \((1, 0, 1)\).

**Definition 7** We call a function \(f(x_1, \ldots, x_n)\) monotone if \(f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n)\) whenever \((x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)\).

**Example 16** The functions \(x_1 \land x_2\) and \(x_2 \lor x_2\) are monotone, however the functions \(\bar{x}_1\) and \(x_1 \oplus x_2\) are not.

Denote by \(M_n\) the number of monotone Boolean functions of \(n\) variables. The problem of computing \(M_n\) was posed by Dedekind in 1897 (!) and is still unsolved up to now. It is known that

<table>
<thead>
<tr>
<th>(n)</th>
<th>(M_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
</tr>
<tr>
<td>4</td>
<td>168</td>
</tr>
<tr>
<td>5</td>
<td>7581</td>
</tr>
<tr>
<td>6</td>
<td>7828354</td>
</tr>
</tbody>
</table>

Many mathematicians contributed to this problem. The most recent to our knowledge result (cf. [1, 4]) is the asymptotic formula for \(M_n\) as \(n \to \infty\):

\[
M_n \sim \begin{cases} 
2^{\left(n/2\right)} \exp \left\{ \left(\sum_{i=1}^{n} 2^{-i/2} \right) \left(2^{-n/2} + n^2 \cdot 2^{-n-5} - n \cdot 2^{-n-4}\right) \right\}, & \text{if } n \text{ is even} \\
2 \cdot 2^{\left(n^2 / 2\right)} \exp \left\{ \left(\sum_{i=1}^{n+1} 2^{-i/2} \right) \left(2^{-(n+1)/2} + n^2 \cdot 2^{-n-4}\right) + \left(\sum_{i=1}^{n-3} 2^{-i/2} \right) \left(2^{-(n+3)/2} - n^2 \cdot 2^{-n-6} - n \cdot 2^{-n-3}\right) \right\}, & \text{if } n \text{ is odd.}
\end{cases}
\]

7 The criterion for completeness

Let \(\Sigma = \{f_1, f_2, \ldots, f_i, \ldots\}\) be a set of Boolean functions.
**Theorem 1** (E. Post [2, 3])

The set $\Sigma$ is complete if and only if for any of the classes $T_0, T_1, L, S, M$ there exists a function of $\Sigma$ which is not in this class.

In order to apply this theorem to the set $\Sigma$ we construct the following table:

<table>
<thead>
<tr>
<th></th>
<th>$T_0$</th>
<th>$T_1$</th>
<th>$L$</th>
<th>$S$</th>
<th>$M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\ldots$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_i$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\ldots$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

with entries of the set $\{+, -\}$. The entry “+” in the $i^{th}$ row means that the function $f_i$ belongs to the corresponding class. Then, by the theorem of Post, the set $\Sigma$ is complete if and only if each column of this table contains at least one “-”.

**Example 17** Consider the system $\Sigma_1 = \{\overline{x}_1, x_1 \lor x_2, x_1 \land x_2\}$. One has:

<table>
<thead>
<tr>
<th></th>
<th>$T_0$</th>
<th>$T_1$</th>
<th>$L$</th>
<th>$S$</th>
<th>$M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\overline{x}_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_1 \lor x_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_1 \land x_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Thus, by the theorem of Post the set $\Sigma$ is complete.

Moreover, one of the last two functions (but not both) can be deleted from $\Sigma$ without the lost of the completeness of the remaining set. In such a way the complete system $\Sigma_2$ of Example 2 can be obtained.

**Example 18** Consider the following set $\Sigma$:

$f_1 = x_1 x_2$, $f_2 = 0$, $f_3 = 1$, $f_4 = x_1 \oplus x_2 \oplus x_3$.

One has:

<table>
<thead>
<tr>
<th></th>
<th>$T_0$</th>
<th>$T_1$</th>
<th>$L$</th>
<th>$S$</th>
<th>$M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_4$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Thus, $\Sigma$ is complete. However, deleting of any function from $\sigma$ makes the remaining set incomplete because

$\{f_2, f_3, f_4\} \subset L$ \quad $\{f_1, f_3, f_4\} \subset T_1$$\{f_1, f_2, f_4\} \subset T_0$ \quad $\{f_1, f_2, f_3\} \subset M$

**Corollary 1** Any complete set $\Sigma$ of functions contains a complete subset consisting of at most 5 functions of $\Sigma$. 

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In fact a more strong result holds: any complete set can be reduced to a complete subset consisting of at most 4 functions. As Example 18 shows, this proposition cannot be further improved in general.

References


