On Antichains in Product Posets

Sergei L. Bezrukov
Department of Math and Computer Science
University of Wisconsin - Superior
Superior, WI 54880, USA
sb@mcs.uwsuper.edu

Ian T. Roberts
School of Engineering and Logistics
Charles Darwin University
Darwin, NT 0909, Australia
ian.roberts@cdu.edu.au

Abstract

A corollary of Hilbert’s basis theorem is that any antichain in the set of $n$-dimensional vectors with non-negative entries is finite. In other words, any antichain in the poset given by cartesian powers of semi-infinite chains is finite. We study several variations of this result. We provide necessary and sufficient conditions for antichains in the cartesian product of posets to be finite or bounded in size. Corresponding results are obtained for the rank-difference of antichains in ranked posets.

1 Introduction

Let $(P, \preceq_P)$ be a poset with element set $P$ and partial order $\preceq_P$. Usually we denote the poset and its element set by the same letter.

**Definition 1** The cartesian product of posets $(P, \preceq_P)$ and $(Q, \preceq_Q)$ is the poset with element set $P \times Q$ and the partial order $\preceq_{PQ}$ defined as follows: $(x, y) \preceq_{PQ} (x', y')$ iff $x \preceq_P x'$ and $y \preceq_Q y'$.

Since the cartesian product operation is associative, the cartesian powers $P^n$ are well defined for $n \geq 2$.

**Definition 2** Let $(P, \preceq_P)$ be a poset. A set of elements $S \subseteq P$ is called a chain if $S$ is totally ordered. That is, for any $x, y \in S$ either $x \preceq_P y$ or $y \preceq_P x$. 

1
If $S = P$ in the last definition, the poset itself is called a chain poset. The chain poset with elements given by the natural numbers naturally ordered is denoted by $C_{\geq 0}$.

**Definition 3** A set of elements $A \subseteq P$ is called an antichain if for any two distinct elements $x, y \in A$ neither $x \leq_P y$ nor $y \leq_P x$ holds. The cardinality of a maximum-size antichain of $P$ (if it exists) is called the width of $P$ and denoted by $\text{wd}(P)$. Otherwise, we say that $\text{wd}(P)$ is infinite.

Hilbert’s basis theorem[3], established by David Hilbert in 1888, implies that if $K$ is a field then every ideal in the ring of multivariate polynomials $K[x_1, x_2, \ldots, x_n]$ is finitely generated. A corollary of this result applied to the ideals generated by the monomials over the field $\{0, 1\}$ can be translated into poset terms as follows: every antichain of $C^n_{\geq 0}$ is finite for any $n \geq 2$.

We study a similar question for other infinite posets and their cartesian products. In Section 2 we provide necessary and sufficient conditions on $P$ and $Q$ to guarantee that antichains in $P \times Q$ are finite or bounded in size. Section 3 extends the results to heights of antichains in ranked posets. The concluding section includes $k$-antichains and an open question.

## 2 Posets

In the next theorem we assume that both $\text{wd}(P)$ and $\text{wd}(Q)$ are finite. This condition is obviously necessary for every antichain of $P \times Q$ to be finite or bounded in size.

For an element $x \in P$ denote by $\bar{x}$ a maximum size chain in $P$ for which $x$ is the maximum element. Similarly, for $x \in P$ denote by $\bar{y}$ a maximum size chain in $P$ for which $x$ is the minimum element. If several chains exist for a given $x$, we pick any one of them. The cardinalities $|\bar{x}|$ or $|\bar{y}|$ can be infinite.

**Theorem 1** For any posets $P, Q$, each with finite width, the poset $P \times Q$ contains an infinite antichain iff $|\bar{x}|$ of $|\bar{y}|$ is infinite for some $x \in P$ and $y \in Q$ respectively is infinite for some $y \in Q$.

**Proof.**

To show the sufficiency, assume both $|\bar{x}|$ and $|\bar{y}|$ are infinite for some $x \in P$ and $y \in Q$. Let $\bar{x} = \{x, x_1, x_2, \ldots\}$ and $\bar{y} = \{y, y_1, y_2, \ldots\}$ where

$$\cdots \prec_P x_3 \prec_P x_2 \prec_P x_1 \prec_P x \prec_P y \prec_Q y_1 \prec_Q y_2 \prec_Q y_3 \prec_Q \cdots$$

Then we have the following infinite antichain in $P \times Q$:

$$\{(x, y), (x_1, y_1), (x_2, y_2), \ldots, (x_i, y_i), \ldots\}.$$ 

To prove the necessity of the condition, suppose $A \subseteq P \times Q$ is an infinite antichain and $|\bar{x}|$ is finite for every $x \in P$, and let

$$P' = \{x \in P \mid y \in Q \text{ and } (x, y) \in A\}.$$
Consider the subposet of $P$ formed by the element set $P'$ and the induced partial order on $P$. Note that for every fixed $x \in P$ the set $\{y \in Q \mid (x, y) \in A\}$ is an finite antichain in $Q$ as $\text{wd}(Q)$ is finite. Therefore, as $A$ is infinite, $P'$ must also be infinite. Since $P' \subseteq P$, $\text{wd}(P')$ is finite. Hence, by Dilworth’s theorem [1], $P'$ can be partitioned into a finite number of chains. Since $P'$ is infinite, and $|x|$ is finite for any $x \in P$, $P'$ contains an infinite increasing chain $x_1 \prec_P x_2 \prec_P \cdots$, which we denote by $C$. Let

$$Q' = \{y \in Q \mid x \in C \quad \text{and} \quad (x, y) \in A\}.$$ 

Since $C$ is infinite, $Q'$ is infinite. Consider the subposet of $Q$ formed by the element set $Q'$ and the induced partial order on $Q$. Since $Q' \subseteq Q$, $\text{wd}(Q')$ is finite. Applying Dilworth’s theorem again, we conclude that $Q'$ must contain an infinite chain $D$. Since $A$ is an antichain and $C$ is an infinite increasing chain, $D$ must be decreasing. So there exists $y \in Q$ with an infinite $|y|$. To complete the proof note that if $x_1 \in P$ as constructed above then $|x_1|$ is infinite. □

Although the size of each antichain in $P \times Q$ is finite, the set of such sizes may be arbitrarily large, and hence $\text{wd}(P \times Q)$ is infinite. For example, if $P = Q = C_{\geq 0}$ then for any fixed $k \geq 0$ the set $\{(x, y) \mid x + y = k\}$ is a finite antichain in $P \times Q$ but the set of such sizes becomes arbitrarily large as $k$ grows.

**Corollary 1** For any $n \geq 2$ and poset $P$ with finite width, $P^n$ contains an infinite antichain, iff $|x|$ is infinite for some $x \in P$ and $|y|$ is infinite for some $y \in P$.

In the next theorem we determine when a maximum-size antichain in $P \times Q$ is finite. Note that for any antichain $A \subseteq P$ and every fixed $y \in Q$ the set $A \times \{y\}$ is an antichain in $P \times Q$. So, if every antichain in $P \times Q$ is finite, then $\text{wd}(P)$ must be finite. By a similar reason $\text{wd}(Q)$ must be finite.

**Theorem 2** For any posets $P, Q$, each with finite width, $\text{wd}(P \times Q)$ is finite iff $P$ or $Q$ is finite.

**Proof.**

To prove the necessity, we prove the contrapositive. Assume both $P$ and $Q$ are infinite. Since $\text{wd}(Q)$ is finite, Dilworth’s theorem implies that $Q$ contains an infinite chain $C = \{y_1, y_2, \ldots\}$. Without loss of generality we assume $C$ is increasing, so $y_1 \prec_Q y_2 \prec_Q \cdots$. By a similar reason $P$ must contain an infinite chain. If this chain is increasing, $x_1 \prec_P x_2 \prec_P \cdots$, then for every fixed $k \geq 2$ the set $\{(x_i, y_j) \mid i + j = k\}$ is an antichain in $P \times Q$ and becomes arbitrary large as $k$ increases. Otherwise, if $P$ contains an infinite decreasing chain $x_1 \succ_P x_2 \succ_P \cdots$, then the set $\{(x_i, y_i) \mid i \geq 1\}$ is an infinite antichain in $P \times Q$. In either case $\text{wd}(P \times Q)$ is infinite.

To show the sufficiency, assume $P$ is finite. Let $A$ be an antichain in $P \times Q$. Since for every fixed $x \in P$ the set $\{y \in Q \mid x \in P \quad \text{and} \quad (x, y) \in A\}$ is an antichain in $Q$. As $|A| \leq |P| \cdot \text{wd}(Q)$, $\text{wd}(P \times Q)$ is finite. □

Note that if $A_P \subseteq P$ and $A_Q \subseteq Q$ are antichains, then $A_P \times A_Q$ is an antichain in $P \times Q$. This implies the lower bound in

$$\text{wd}(P) \cdot \text{wd}(Q) \leq \text{wd}(P \times Q) \leq \min\{|P| \cdot \text{wd}(Q), \quad |Q| \cdot \text{wd}(P)|.$$
The upper bound follows from the proof of Theorem 2. Both bounds are attainable if \( P \) or \( Q \) is an antichain.

If both \( P \) and \( Q \) are infinite, \( \text{wd}(P \times Q) \) might be infinite, as the example \( P = Q = C_{\geq 0} \) shows.

**Corollary 2** For any \( n \geq 2 \), \( \text{wd}(P^n) \) is finite iff \( P \) is finite.

## 3 Ranked Posets

**Definition 4** A poset \( (P, \preceq_P) \) is called ranked if there exists a rank function \( r_P : P \to \mathbb{N} \) (\( \mathbb{N} \) being the set of natural numbers) such that \( \min_{x \in P} r_P(x) = 0 \) and \( r_P(x) = r_P(y) - 1 \) for every pair \( x \prec_P y \) such that there is no \( z \in P \) with \( x \prec_P z \prec_P y \).

The cartesian product of ranked posets is also a ranked poset and \( r_{P \times Q}(x, y) = r_P(x) + r_Q(y) \).

Let \( P_k = \{x \in P \mid r_P(x) = k\} \) denote the \( k \)-th level of \( P \), and let \( r(P) = \max_{x \in P} r_P(x) \) denote the rank of \( P \).

**Definition 5** For a ranked poset \( (P, \preceq_P) \) and a subset \( A \subseteq P \), we call the magnitude \( \text{rd}(A) = \max_{x \in A} r_P(x) - \min_{x \in A} r_P(x) \) the rank-difference of \( A \). Let \( \text{rd}(P) = \max_A \text{rd}(A) \), where the maximum runs over all antichains of \( P \). The values \( \text{rd}(A) \) and \( \text{rd}(P) \) can be infinite.

**Theorem 3** For ranked posets \( P \) and \( Q \), \( \text{rd}(P \times Q) \) is finite iff both \( r(P) \) and \( r(Q) \) are finite, or \( |Q| = 1 \) and \( \text{rd}(P) \) is finite, or \( |P| = 1 \) and \( \text{rd}(Q) \) is finite.

**Proof.**

To prove the sufficiency, if both \( r(P) \) and \( r(Q) \) are finite, then \( r(P \times Q) = r(P) + r(Q) \) is finite, which implies \( \text{rd}(P \times Q) \) is finite. If \( |Q| = 1 \) and \( r(P) \) is infinite then every antichain in \( P \times Q \) is of the form \( A \times Q \) with \( A \) being an antichain in \( P \). Then \( \text{rd}(P \times Q) = \text{rd}(P) \).

For the necessity of the condition assume \( \text{rd}(P \times Q) \) is finite but one of the posets, say \( P \), has an infinite rank. So, \( P \) contains an infinite increasing chain \( x_1 \prec_P x_2 \prec_P \cdots \). Avoiding the trivial case, assume \( |Q| > 1 \). Let \( y_1, y_2 \in Q \) with \( r_Q(y_1) \leq r_Q(y_2) \). Then for every \( k \geq 1 \) the set \( A = \{(x_1, y_2), (x_k, y_1)\} \) is an antichain in \( P \times Q \). One has \( \text{rd}(A) = (r_P(x_k) - r_P(x_1)) + (r_Q(y_2) - r_Q(y_1)) \geq k \), which becomes arbitrary large as \( k \) increases, contradicting the finite value of \( \text{rd}(P \times Q) \). \( \square \)

Note that
\[
\text{rd}(P) + \text{rd}(Q) \leq \text{rd}(P \times Q) \leq \max\{\text{rd}(P) + r(Q), \text{rd}(Q) + r(P)\}.
\]

To show the lower bound, let \( A_P \subseteq P \) be an antichain with \( \text{rd}(A_P) = \text{rd}(P) \). Similarly, let \( A_Q \subseteq Q \) be an antichain with \( \text{rd}(A_Q) = \text{rd}(Q) \). Then \( A^* = A_P \times A_Q \) is an antichain in \( P \times Q \).

One has
\[
\text{rd}(P \times Q) = \max_{(x,y) \in A} \max_{(x,y) \in A} r_{P \times Q}((x,y)) - \min_{(x,y) \in A} r_{P \times Q}((x,y))
\]
If \( r \) finite (by the (\( P, Q \))-condition and (\( P, Q \))-condition), but \( P \times Q \) has an infinite rank-difference, then for any \( A \subseteq P \times Q \), has an infinite rank-difference.

\[
\begin{align*}
\geq & \max_{(x,y) \in A^*} r_{P \times Q}((x,y)) - \min_{(x,y) \in A^*} r_{P \times Q}((x,y)) \\
= & \max_{(x,y) \in A^*} (r_P(x) + r_Q(y)) - \min_{(x,y) \in A^*} (r_P(x) + r_Q(y)) \\
= & (\max_{x \in A_P} r_P(x) - \min_{x \in A_P} r_P(x)) + (\max_{y \in A_Q} r_Q(y) - \min_{y \in A_Q} r_Q(y)) \\
= & \text{rd}(P) + \text{rd}(Q).
\end{align*}
\]

For the upper bound, let \( A \subseteq P \times Q \) be an antichain with the maximum rank-difference, and let \( (x_1, y_1), (x_2, y_2) \in A \). If \( \{x_1, x_2\} \) is an antichain in \( P \), then \( r_P(x_1) - r_P(x_2) \leq \text{rd}(P) \). In this case \( y_1, y_2 \) can be arbitrary, so \( r_Q(y_1) - r_Q(y_2) \leq \text{rd}(Q) \). Otherwise, if \( x_1 \not<_{P} x_2 \) and \( y_2 \not<_{Q} y_1 \) then \( r_{P \times Q}((x_1, y_1)) - r_{P \times Q}((x_2, y_2)) \leq \max\{\text{rd}(P), \text{rd}(Q)\} - 1 \). The remaining cases can be handled similarly.

The bounds are attainable if \( P \) or \( Q \) is an antichain. The upper bound is also attainable if \( P \) and \( Q \) are the Boolean lattices.

**Corollary 3** For any \( n \geq 2 \), \( \text{rd}(P^n) \) is finite iff \( r(P) \) is finite.

In the next theorem we assume that the rank-difference of every antichain in \( P \) and \( Q \) is finite. This is necessary for the rank-difference of any antichain of \( P \times Q \) to be finite. Indeed, if there is an antichain \( A \subseteq P \) with an infinite rank-difference, then for any \( y \in Q \), the set \( A \times \{y\} \) is an antichain in \( P \times Q \) with an infinite rank-difference.

For a pair of posets \( (P, Q) \) we introduce the \((P, Q)\)-condition: if \( r(P) \) is infinite then \( \text{wd}(Q) \) is finite.

**Theorem 4** Let \( P \) and \( Q \) be ranked posets with every antichain having a finite rank-difference. Every antichain in \( P \times Q \) has a finite rank-difference iff either both \( r(P) \) and \( r(Q) \) are finite, or both the \((P, Q)\)-condition and \((Q, P)\)-condition are satisfied.

**Proof.**
To prove the necessity, assume every antichain in \( P \times Q \) has a finite rank-difference and one of the posets, say \( P \), has an infinite rank. Then, \( P \) contains an infinite increasing chain \( x_1 \not<_{P} x_2 \not<_{P} \cdots \). If \( Q \) has an infinite antichain \( \{y_1, y_2, \ldots\} \) with \( r_Q(y_1) \leq r_Q(y_2) \leq \cdots \), then the following antichain in \( P \times Q \) has an infinite rank-difference:

\[
\{(x_1, y_1), (x_2, y_2), \ldots, (x_i, y_i), \ldots\}.
\]

To prove the sufficiency of the above condition, we omit a trivial case when both \( r(P) \) and \( r(Q) \) are finite. Assume every antichain in \( P \) has a finite rank-difference, \( r(P) \) is infinite, \( \text{wd}(Q) \) is finite (by the \((P, Q)\)-condition), but \( P \times Q \) has an antichain \( A \) with an infinite rank-difference. If \( r(Q) \) is infinite, \( \text{wd}(P) \) is finite by the \((Q, P)\)-condition. Theorem 1 implies every antichain in \( P \times Q \) is finite, hence has a finite rank-difference.

So, we can assume \( r(Q) \) is finite. Since \( \text{wd}(Q) \) is finite, Dilworth’s theorem implies \( Q \) is finite. Denote by \( P' \subseteq P \) and \( Q' \subseteq Q \) the subposets with the element sets respectively \( \{x \in P \mid y \in Q \text{ and } (x, y) \in A\} \) and \( \{y \in Q \mid x \in P \text{ and } (x, y) \in A\} \) and the induced partial orders.
Note that \( r(P') \) is infinite, since otherwise \( \mathcal{A} \subseteq P' \times Q' \) and \( \text{rd}(P' \times Q') \leq r(P' \times Q') = r(P') + r(Q') \) would imply \( \mathcal{A} \) has a finite rank-difference. Hence, \( P' \) contains an infinite increasing chain \( x_1 \prec_p x_2 \prec_p \cdots \), such that \( \{(x_1, y_1), (x_2, y_2), \ldots \} \subseteq \mathcal{A} \) for some \( y_1, y_2, \ldots \in Q' \). Since \( Q' \subseteq Q \) is finite, some element \( y_i \in Q' \) appears twice in the above set of pairs. This contradicts the fact that \( \mathcal{A} \) is an antichain and completes the proof.

**Corollary 4** For any \( n \geq 2 \), every antichain in \( P^n \) has a finite rank-difference iff either \( r(P) \) is finite or every antichain in \( P \) has a finite rank-difference and \( \text{wd}(P) \) is finite.

### 4 Concluding remarks

**Open Question**

Suppose \( P \) is an infinite ranked poset and \( \max_k |P_k| \) is finite. What minimal set of conditions can be added to \( P \) to ensure that every antichain is finite? What minimal set of conditions can be added to \( P \) to ensure that \( \text{wd}(P) \) is finite?

The size of every level of the infinite ranked poset in Fig. 1 is at most 3, however, it contains an infinite antichain shown by empty circles.

\[ \text{Figure 1: A ranked poset with an infinite antichain} \]

An immediate extension of our results involves \( k \)-antichains.

**Definition 6** A set \( \mathcal{A} \subseteq P \) is called a \( k \)-antichain if it does not contain a chain of length \( k + 1 \). Thus, usual antichains are 1-antichains. (See [2]).

**Lemma 1** (see Lemma 4.3.1 in [2])

If \( \mathcal{A} \subseteq P \) is a \( k \)-antichain, then there exists \( k \) pairwise disjoint antichains \( \mathcal{A}_1, \ldots, \mathcal{A}_k \) such that \( \mathcal{A} = \bigcup_{i=1}^k \mathcal{A}_i \). Conversely, the union of \( k \) pairwise disjoint antichains is a \( k \)-antichain.

The above lemma shows that dealing with \( k \)-antichains is, in a sense, equivalent to dealing with antichains.

**Corollary 5** For any fixed \( k \geq 1 \) any \( k \)-antichain of \( P \) is finite iff every antichain of \( P \) is finite.
Corollary 6 For any fixed $k \geq 1$ any $k$-antichain of $P$ has a finite rank-difference iff it is so for every antichain of $P$.

This allows a reformulation of Theorems 1 – 4 for $k$-antichains.

Acknowledgments

The authors are grateful to two anonymous referees for their constructive comments that significantly improved the quality of the paper.

References

