The Cyclic Wirelength of Trees

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Abstract

We consider the problem of embedding trees into cyclic host graphs in such a way as to minimize the wirelength. We show that the cyclic wirelength of trees equals their linear wirelength.

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Denote by $T = (V_T, E_T)$, $C = (V_C, E_C)$ and $L = (V_L, E_L)$ respectively a tree, a cycle and a path with $n$ vertices. Following \cite{1}, define for a graph $H = (V_H, E_H)$ with $|V_H| = n$ a $H$-layout of $T$ as an ordered pair $(\pi, P_\pi)$ consisting of a bijective mapping $\pi : V_T \mapsto V_H$ and a collection $P_\pi$ of paths in $H$, one path joining $\pi(v)$ with $\pi(w)$ for each pair of adjacent vertices $v$ and $w$ of $V_T$. For $H \in \{C, L\}$ and $e \in E_H$ let

\[ con_H(e) = |\{p \in P_\pi | e \in p\}|, \]

be the congestion of the edge $e$ in the layout $(\pi, P_\pi)$. Furthermore, let

\[ cwl(T) = \min_\pi \sum_{e \in E_C} con_C(e) \quad \text{and} \quad lwl(T) = \min_\pi \sum_{e' \in E_L} con_L(e') \]

be the cyclic and the linear wirelength of the tree $T$ respectively.

**Theorem 1** For any tree $T$, $cwl(T) = lwl(T)$.

Proof.

Obviously, $cwl(T) \leq lwl(T)$. We show the inverse inequality. Therefore, we consider a one-to-one embedding $\pi_C$ of $T$ into $C$ and transform it, as it is described in \cite{1}, into an one-to-one embedding $\pi_L$ of $T$ into $L$. Denote by $f : V_C \mapsto V_L$ the bijective mapping induced by this transformation.

We denote the vertices of $L$ by $l_1, l_2, \ldots, l_n$ counted from one end to the other. Let $e \in E_T$ and let a path $p_e$ be its image in embedding $\pi_C$. For $i = 1, \ldots, n-1$
and \((l_i, l_{i+1}) \in E_L\) denote \(R_i = \{p_e \mid e \in E_T, (l_i, l_{i+1}) \in \pi_L(e)\}\).

In other words, \(R_i\) is the set of all paths \(p_e\), which contain the edge \((l_i, l_{i+1})\) in the embedding \(\pi_L\). Clearly,
\[
\text{con}_L((l_i, l_{i+1})) = |R_i|.
\] (1)

Now for \((l_i, l_{i+1}) \in E_L\) denote \(v = f^{-1}(l_i)\) and \(w = f^{-1}(l_{i+1})\). The vertices \(v\) and \(w\) partition the cycle \(C\) into two paths \(p_1\) and \(p_2\) (cf. [1]).

**Lemma 1** (see [1], proof of Theorem 1). Let \((l_i, l_{i+1}) \in E_L\). Then either \(p_1\) is a subpath of \(p\) for any path \(p \in R_i\) or \(p_2\) is a subpath of \(p\) for any path \(p \in R_i\).

For \((l_i, l_{i+1}) \in E_L\) denote by \(P_i\) the path \(p_1\) or \(p_2\) of \(C\) depending on the situation which alternative of Lemma 1 performs. Let \(E_i\) be the set of edges of the path \(P_i\). Lemma 1 and (1) imply that
\[
\text{con}_C(e) \geq \text{con}_L((l_i, l_{i+1})), \quad \text{for any } e \in E_i.
\] (2)

Assume that we can show that for any \(i = 1, \ldots, n - 1\) there exists an edge \(e_i \in E_i\) so that all the edges \(e_1, \ldots, e_{n-1}\) are distinct. Then, using (2) one has
\[
\sum_{e \in E_C} \text{con}_C(e) \geq \sum_{i=1}^{n-1} \text{con}_C(e_i) \geq \sum_{i=1}^{n-1} \text{con}_L((l_i, l_{i+1})) = \sum_{e' \in E_L} \text{con}_L(e').
\] (3)

Thus, it remains to prove the existence of the System of Distinct Representatives (SDR) of the set system \(\{E_1, \ldots, E_{n-1}\}\). By a theorem of Hall (see e.g. [3]), the SDR exists iff
\[
\left| \bigcup_{i \in I} E_i \right| \geq |I|
\] (4)
for any subset \(I \subseteq \{1, \ldots, n - 1\}\).

To show that the condition (4) is satisfied, consider a sample \(E_{i_1}, \ldots, E_{i_k}\) for some \(k < n\). Now we define a graph \(G\) with \(k\) vertices corresponding to the paths \(P_{i_j}, j = 1, \ldots, k\). Two vertices of \(G\) are adjacent iff the corresponding paths have a common vertex (of \(C\)). Let \(A_1, \ldots, A_t\) be the connected components of the graph \(G\). Assume the component \(A_i\) has \(k_i\) vertices and let us denote by \(P_{i_1}^1, \ldots, P_{i_k}^k\) the paths of \(C\) corresponding to the vertices of \(A_i\). One has \(k_1 + \cdots + k_t = k\).
Consider a component $A_i$ and let $T_i$ be the union of the paths $P_{ij}^i$, $j = 1, \ldots, k_i$. Here by the union of two paths we mean a subgraph of $C$ induced by the edge sets of these paths. Obviously, either $T_i$ is a path of $C$ or $T_i$ is $C$.

**Case 1.** Let $T_i$ be a path of $C$. We know that $T_i$ is the union of $k_i$ distinct paths $P_{ij}^i$. The image of these paths in the mapping $f$ is a collection of $k_i$ distinct edges of $L$. Obviously, the endpoints of these edges form a set consisting of at least $k_i + 1$ vertices of $L$. Since the mapping $f$ is a bijection, the endpoints of the paths $P_{ij}^i$ form a set which also consists of at least $k_i + 1$ vertices of $T_i$. Therefore, the path $T_i$ has at least $k_i + 1$ vertices, and thus, at least $k_i$ edges. Since this holds for any $i = 1, \ldots, t$, and the paths $T_i$ corresponding to different components do not have common vertices, the total number of edges in the union $E_{i_1} \cup \cdots \cup E_{i_k}$ is at least $k_1 + \cdots + k_t = k$, and the condition (4) is satisfied.

**Case 2.** If $T_i$ is the whole cycle $C$, then the number of edges in the union $E_{i_1} \cup \cdots \cup E_{i_k}$ is $n$, and (4) is satisfied as well because $|I| = k < n$. □

Therefore, the results of [1,2] and Theorem 1 respectively imply that for any tree $T$, the corresponding minimal values of congestion, dilation and their minimal average values are the same for embeddings of $T$ into a path and a cycle.

References

