Vectors in 2-Space and 3-Space
Chapter Contents

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1 Introduction to Vectors

(Geometric)
Geometric Vectors

Symbolically, we shall denote vectors in lowercase boldface type. All our scalars will be real numbers and will be denoted in lowercase italic type.

- The vector of length zero is called the zero vector and is denoted by \( \mathbf{0} \).
- Since there is no natural direction for the zero vector, the negative of \( \mathbf{v} \), is defined to be the vector having the same magnitude as \( \mathbf{v} \), but oppositely directed.
If \( \mathbf{v} \) and \( \mathbf{w} \) are any two vectors, then the sum \( \mathbf{v} + \mathbf{w} \) is the vector determined as follows: Position the vector \( \mathbf{w} \) so that its initial point coincides with the terminal point of \( \mathbf{v} \). The vector \( \mathbf{v} + \mathbf{w} \) is represented by the arrow from the initial point of \( \mathbf{v} \) to the terminal point of \( \mathbf{w} \).
Definition

If \( \mathbf{v} \) and \( \mathbf{w} \) are any two vectors, then the difference of \( \mathbf{w} \) from \( \mathbf{v} \) is defined by

\[
\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})
\]
Definition

If \( \mathbf{v} \) is a nonzero vector and \( k \) is a nonzero real number (scalar), then the product \( k\mathbf{v} \) is defined to be the vector whose length is \( |k| \) times the length of \( \mathbf{v} \) and whose direction is the same as that of \( \mathbf{v} \) if \( k > 0 \) and opposite to that of \( \mathbf{v} \) if \( k < 0 \). We define \( k\mathbf{v} = \mathbf{0} \) if \( k = 0 \) or \( \mathbf{v} = \mathbf{0} \).

A vector of the form \( k\mathbf{v} \) is called a scalar multiple.
In Figure 3.1.6, that \( \mathbf{v} \) has been positioned so its initial point is at the origin of a rectangular coordinate system. The coordinates \( v_1, v_2 \) of the terminal point of \( \mathbf{v} \) are called the components of \( \mathbf{v} \), and we write

\[
\mathbf{v} = (v_1, v_2)
\]
If \( \mathbf{v} = (v_1, v_2) \) and \( \mathbf{w} = (w_1, w_2) \), two vectors are equivalent if and only if \( v_1 = w_1 \) and \( v_2 = w_2 \), and

\[
\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2)
\]

\[
k\mathbf{v} = (kv_1, kv_2)
\]

\[
\mathbf{v} - \mathbf{w} = (v_1 - w_1, v_2 - w_2)
\]
Each pair of coordinate axes determines a plane called a coordinate plane. These are referred to as the *xy-plane*, the *xz-plane*, and the *yz-plane*.

To each point $P$ in 3-space we assign a triple of numbers $(x, y, z)$, called the coordinates of $P$. 
Rectangular coordinate systems in 3-space fall into two categories, left-handed and right-handed.

In this book we shall use only right-handed coordinate systems.
A vector \( \mathbf{v} \) in 3-space is positioned so its initial point is at the origin of a rectangular coordinate system. The coordinates of the terminal point of \( \mathbf{v} \) are called the components of \( \mathbf{v} \), and we write \( \mathbf{v} = (v_1, v_2, v_3) \).

If \( \mathbf{v} = (v_1, v_2, v_3) \) and \( \mathbf{w} = (w_1, w_2, w_3) \) are two vectors in 3-space, then

\[
\mathbf{v} \text{ and } \mathbf{w} \text{ are equivalent if and only if } v_1 = w_1, v_2 = w_2, v_3 = w_3
\]

\[
\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3)
\]

\[
k \mathbf{v} = (kv_1, kv_2, kv_3), \text{ where } k \text{ is any scalar}
\]
Sometimes a vector is positioned so that its initial point is not at the origin.

If the vector $\overrightarrow{P_1P_2}$ has initial point $P_1 = (x_1, y_1, z_1)$ and terminal point $P_2(x_2, y_2, z_2)$, then

$$\overrightarrow{P_1P_2} = (x_2, y_2, z_2) - (x_1, y_1, z_1) = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

In 2-space the vector with initial point $P_1(x_1, y_1)$ and terminal point $P_1(x_1, y_1)$ is

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)$$
Example 1

Vector Computations with Components

If \( \mathbf{v} = (1, -3, 2) \) and \( \mathbf{w} = (4, 2, 1) \), then

\[
\mathbf{v} + \mathbf{w} = (5, -1, 3), \quad 2\mathbf{v} = (2, -6, 4) \quad -\mathbf{w} = (-4, -2, -1), \\
\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}) = (-3, -5, 1)
\]
Example 2
Finding the components of a Vector

The components of the vector $v = \overrightarrow{P_1P_2}$ with initial point $P_1 = (2,-1,4)$ and terminal point $P_2(7,5,-8)$ are
$$v = (7 - 2, 5 - (-1), (-8) - 4) = (5, 6, -12)$$
Translation of Axes

In Figure 3.1.14a we have translated the axes of an xy-coordinate system to obtain an \( x'y' \)-coordinate system whose \( O' \) is at point \((x',y') = (k, l)\).

A point \( P \) in 2-space now has both \((x,y)\) coordinates and \((x',y')\) coordinates.

\[ x' = x - k, \quad y' = y - l, \] these formulas are called the translation equations.

In 3-space the translation equations are \( x' = x - k, \) \( y' = y - l, \) \( z' = z - m \) where \((k, l, m)\) are the xyz-coordinates of the \( x'y'z' \)-origin.

\[ \text{Figure 3.1.14} \]
Example 3
Using the Translation Equations (1/2)

Suppose that an xy-coordinate system is translated to obtain an x'y'-coordinate system whose origin has xy-coordinates \((k, l) = (4, 1)\).

(a) Find the x'y'-coordinate of the point with the xy-coordinates \(P(2, 0)\)
(b) Find the xy-coordinate of the point with the x'y'-coordinates \(Q(-1, 5)\)
Example 3
Using the Translation Equations (2/2)

- **Solution (a).** The translation equations are
  \[ x' = x - 4, \quad y' = y - 1 \]
  so the \(x'y'\)-coordinate of \(P(2,0)\) are \(x' = 2 - 4 = -2\) and \(y' = 0 - 1 = -1\).

- **Solution (b).** The translation equations in (a) can be written as
  \[ x = x' + 4, \quad y = y' + 1 \]
  so the \(xy\)-coordinate of \(Q\) are \(x = -1 + 4 = 3\) and \(y = 5 + 1 = 6\).
2 Norm of a Vector; Vector Arithmetic
Theorem 2.1
Properties of Vector Arithmetic

- If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in 2- or 3-space and $k$ and $l$ are scalars, then the following relationship holds:

  (a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  (b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
  (c) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
  (d) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
  (e) $k(l\mathbf{u}) = (kl)\mathbf{u}$
  (f) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
  (g) $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$
  (h) $1\mathbf{u} = \mathbf{u}$
Norm of a Vector (1/2)

- The **length** of a vector $\mathbf{u}$ is often called the **norm** of $\mathbf{u}$ and is denoted by $\|\mathbf{u}\|$.

- Figure (a): it follows from the Theorem of Pythagoras that the norm of a vector $\mathbf{u} = (u_1, u_2)$ in 2-space is $\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2}$.

- Figure (b): Let $\mathbf{u} = (u_1, u_2, u_3)$ be a vector in 3-space.

\[
\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}
\]

- A vector of norm 1 is called a **unit vector**.
Norm of a Vector (2/2)

- If \( P_1 = (x_1, y_1, z_1) \) and \( P_2(x_2, y_2, z_2) \) are two points in 3-space, then the distance \( s \) between them is the norm of vector \( \overrightarrow{P_1P_2} \):

\[
\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)
\]

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}
\]

- Similarly in 2-space:

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

- the length of the vector \( ku \):

\[
\|ku\| = |k| \|u\|
\]
Example 1
Finding Norm and Distance

The norm of the vector \( u = (-3,2,1) \) is

\[
\|u\| = \sqrt{(-3)^2 + (2)^2 + (1)^2} = \sqrt{14}
\]

The distance \( d \) between the points \( P_1(2,-1,5) \) and \( P_2 = (4,-3,1) \) is

\[
d = \sqrt{(4 - 2)^2 + (-3 + 1)^2 + (1 + 5)^2} = \sqrt{44} = 2\sqrt{11}
\]
3 Dot Product; Projections
The Angle Between Vectors

Let \( \mathbf{u} \) and \( \mathbf{v} \) be two nonzero vectors in 2-space or 3-space, and assume these vectors have been positioned so their initial points coincided. By the angle between \( \mathbf{u} \) and \( \mathbf{v} \), we shall mean the angle \( \theta \) determined by \( \mathbf{u} \) and \( \mathbf{v} \) that satisfies \( 0 \leq \theta \leq \pi \).

\[ u \quad \theta \quad v \]

\[ u \quad \theta \quad v \]

\[ u \quad \theta \quad v \]

\[ u \quad \theta \quad v \]

\[ u \quad \theta \quad v \]

**Figure 3.3.1** The angle \( \theta \) between \( \mathbf{u} \) and \( \mathbf{v} \) satisfies \( 0 \leq \theta \leq \pi \).
If \( \mathbf{u} \) and \( \mathbf{v} \) are vectors in 2-space or 3-space and \( \theta \) is the angle between \( \mathbf{u} \) and \( \mathbf{v} \), then the *dot product* or *Euclidean inner product* \( \mathbf{u} \cdot \mathbf{v} \) is defined by

\[
\mathbf{u} \cdot \mathbf{v} = \begin{cases} 
\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta & \text{if } \mathbf{u} \neq \mathbf{0} \text{ and } \mathbf{v} \neq \mathbf{0} \\
0 & \text{if } \mathbf{u} = \mathbf{0} \text{ or } \mathbf{v} = \mathbf{0}
\end{cases}
\] (1)
Example 1

Dot Product

As shown in Figure 3.3.2, the angle between the vectors $\mathbf{u} = (0, 0, 1)$ and $\mathbf{v} = (0, 2, 2)$ is $45^\circ$. Thus,

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| ||\mathbf{v}|| \cos \theta = (\sqrt{0^2 + 0^2 + 1^2})(\sqrt{0^2 + 2^2 + 2^2}) \left( \frac{1}{\sqrt{2}} \right) = 2$$

![Figure 3.3.2](image)
Component Form of the Dot Product (1/2)

Let \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \) be two nonzero vectors. If as shown in figure 3.3.3, \( \theta \) is the angle between \( u \) and \( v \), then the law of cosines yields

\[
\|PQ\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\theta
\]

(2)

Since \( \overrightarrow{PQ} = v - u \), we can rewrite (2) as

\[
\|u\|\|v\|\cos\theta = \frac{1}{2} (\|u\|^2 + \|v\|^2 - \|v - u\|^2)
\]

or

\[
u \cdot v = \frac{1}{2} (\|u\|^2 + \|v\|^2 - \|v - u\|^2)
\]
Component Form of the Dot Product (2/2)

Substituting

\[ \|u\|^2 = u_1^2 + u_2^2 + u_3^2, \quad \|v\|^2 = v_1^2 + v_2^2 + v_3^2 \]

and \[ \|v - u\|^2 = (v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2 \]

we obtain after Simplfying

\[ u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3 \]

Similarly in 2-space:

\[ u \cdot v = u_1 v_1 + u_2 v_2 \]

The formula is also valid if \( u = 0 \) or \( v = 0 \).
Finding the Angle Between Vectors

- If \( \mathbf{u} \) and \( \mathbf{v} \) are nonzero vectors then

\[
\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \quad (1)
\]

it also can be written as

\[
\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}
\]
Example 2
Dot Product Using [3]

Consider the vectors \( \mathbf{u} = (2, -1, 1) \) and \( \mathbf{v} = (1, 1, 2) \). Find \( \mathbf{u} \cdot \mathbf{v} \) and determine the angle \( \theta \) between \( \mathbf{u} \) and \( \mathbf{v} \).

\[ \mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = (2)(1) + (-1)(1) + (1)(2) = 3 \]

For the given vectors we have \( \|\mathbf{u}\| = \|\mathbf{v}\| = \sqrt{6} \), so that from (5)

\[ \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{3}{\sqrt{6} \sqrt{6}} = \frac{1}{2} \]

Thus, \( \theta = 60^\circ \).
Example 3
A Geometric Problem

Find the angle between a diagonal of a cube and one of its edges.

Solution.

Let $k$ be the length of an edge and introduce a coordinate system as shown in Figure 3.3.4. If we let $\mathbf{u}_1 = (k, 0, 0)$, $\mathbf{u}_2 = (0, k, 0)$, and $\mathbf{u}_3 = (0, 0, k)$, then the vector

$$\mathbf{d} = (k, k, k) = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$$

is a diagonal of the cube. The angle $\theta$ between $\mathbf{d}$ and the edge $\mathbf{u}_1$ satisfies

$$\cos \theta = \frac{\mathbf{u}_1 \cdot \mathbf{d}}{\|\mathbf{u}_1\| \|\mathbf{d}\|} = \frac{k^2}{(k)(\sqrt{3}k^2)} = \frac{1}{\sqrt{3}}$$

Thus,

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.74^\circ$$

Figure 3.3.4
Theorem 3.1

Let \( \mathbf{u} \) and \( \mathbf{v} \) be vectors in 2- or 3-space.

(a) \( \mathbf{v} \cdot \mathbf{v} = \| \mathbf{v} \|^2 \); that is, \( \| \mathbf{v} \| = (\mathbf{v} \cdot \mathbf{v})^{1/2} \)

(b) If the vectors \( \mathbf{u} \) and \( \mathbf{v} \) are nonzero and \( \theta \) is the angle between them, then

\[
\begin{align*}
\theta & \text{ is acute} & \text{if and only if} & \mathbf{u} \cdot \mathbf{v} > 0 \\
\theta & \text{ is obtuse} & \text{if and only if} & \mathbf{u} \cdot \mathbf{v} < 0 \\
\theta & = \pi/2 & \text{if and only if} & \mathbf{u} \cdot \mathbf{v} = 0
\end{align*}
\]
Example 4
Finding Dot products from Components

If \( \mathbf{u} = (1, -2, 3) \), \( \mathbf{v} = (-3, 4, 2) \), and \( \mathbf{w} = (3, 6, 3) \), then

\[
\mathbf{u} \cdot \mathbf{v} = (1)(-3) + (-2)(4) + (3)(2) = -5
\]
\[
\mathbf{v} \cdot \mathbf{w} = (-3)(3) + (4)(6) + (2)(3) = 21
\]
\[
\mathbf{u} \cdot \mathbf{w} = (1)(3) + (-2)(6) + (3)(3) = 0
\]

Therefore, \( \mathbf{u} \) and \( \mathbf{v} \) make an obtuse angle, \( \mathbf{v} \) and \( \mathbf{w} \) make an acute angle, and \( \mathbf{u} \) and \( \mathbf{w} \) are perpendicular.
Orthogonal Vectors

- Perpendicular vectors are also called orthogonal vectors.
- In light of Theorem 3.1.1.b, two nonzero vectors are orthogonal if and only if their dot product is zero.
- To indicate that \( \mathbf{u} \) and \( \mathbf{v} \) are orthogonal vectors we write \( \mathbf{u} \perp \mathbf{v} \).
Example 5
A Vector Perpendicular to a Line

Show that in 2-space the nonzero vector \( \mathbf{n} = (a, b) \) is perpendicular to the line \( ax + by + cz = 0 \).

Solution

Let \( P_1(x_1, y_1) \) and \( P_2(x_2, y_2) \) be distinct points on the line, so that
\[
ax_1 + by_1 + c = 0 \\
ax_2 + by_2 + c = 0
\]  
(6)

Since the vector \( \overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1) \) runs along the line (Figure 3.3.5), we need only show that \( \mathbf{n} \) and \( \overrightarrow{P_1P_2} \) are perpendicular. But on subtracting the equations in (6) we obtain
\[
a(x_2 - x_1) + b(y_2 - y_1) = 0
\]
which can be expressed in the form
\[
(a, b) \cdot (x_2 - x_1, y_2 - y_1) = 0 \quad \text{or} \quad n \cdot \overrightarrow{P_1P_2} = 0
\]
Thus, \( \mathbf{n} \) and \( \overrightarrow{P_1P_2} \) are perpendicular.
Theorem 3.2
Properties of the Dot Product

If $u$, $v$ and $w$ are vectors in 2- or 3-space and $k$ is a scalar, then:

(a) $u \cdot v = v \cdot u$
(b) $u \cdot (v + w) = u \cdot v + u \cdot w$
(c) $k(u \cdot v) = (ku) \cdot v = u \cdot (kv)$
(d) $v \cdot v > 0$ if $v \neq 0$, and $v \cdot v = 0$ if $v = 0$
An Orthogonal Projection (1/2)

- To "decompose" a vector $u$ into a sum of two terms, one parallel to a specified nonzero vector $a$ and the other perpendicular to $a$.

- Figure 3.3.6: Drop a perpendicular from the tip of $u$ to the line through $a$, and construct the vector $w_1$ from $Q$.

- Next form the difference: $w_2 = u - w_1$ then $w_1 + w_2 = w_1 + (u - w_1) = u$

**Figure 3.3.6** The vector $u$ is the sum of $w_1$ and $w_2$, where $w_1$ is parallel to $a$ and $w_2$ is perpendicular to $a$. 
An Orthogonal Projection (2/2)

- The vector \( \mathbf{w}_1 \) is called the orthogonal projection of \( \mathbf{u} \) on \( \mathbf{a} \) or sometimes the vector component of \( \mathbf{u} \) along \( \mathbf{a} \). It is denoted by \( \text{proj}_a \mathbf{u} \). (7)

- The vector \( \mathbf{w}_1 \) is called the vector component of \( \mathbf{u} \) orthogonal to \( \mathbf{a} \). Since we have \( w_2 = u \mathbf{a} \mathbf{a}^{-1} \), this vector can be written in notation (7) as

\[
    w_2 = u - \text{proj}_a u
\]
Theorem 3.3

If $u$ and $a$ are vectors in 2-space or 3-space and if $a \neq 0$, then

$$\text{proj}_a u = \frac{u \cdot a}{\|a\|^2} a \quad (\text{vector component of } u \text{ along } a)$$

$$u - \text{proj}_a u = u - \frac{u \cdot a}{\|a\|^2} a \quad (\text{vector component of } u \text{ orthogonal to } a)$$
An Orthogonal Projection (cont)

A formula for the length of the vector component of $\mathbf{u}$ along $\mathbf{a}$ can be obtained by writing

$$ \|\text{proj}_a \mathbf{u}\| = \left\| \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2} \right\| \|\mathbf{a}\| = \frac{\|\mathbf{u} \cdot \mathbf{a}\|}{\|\mathbf{a}\|} $$

which yields

$$ \|\text{proj}_a \mathbf{u}\| = \frac{\|\mathbf{u} \cdot \mathbf{a}\|}{\|\mathbf{a}\|} $$

If $\theta$ denotes the angle between $\mathbf{u}$ and $\mathbf{a}$, then $\mathbf{u} \cdot \mathbf{a} = \|\mathbf{u}\| \|\mathbf{a}\| \cos \theta$, so that (10) can also be written as

$$ \|\text{proj}_a \mathbf{u}\| = \|\mathbf{u}\| \left| \cos \theta \right| $$
Example 6
Vector Component of \( u \) Along \( a \)

Let \( u = (2, -1, 3) \) and \( a = (4, -1, 2) \). Find the vector component of \( u \) along \( a \) and the vector component of \( u \) orthogonal to \( a \).

Solution:

\[
 u \cdot a = (2)(4) + (-1)(-1) + (3)(2) = 15 \\
\|a\|^2 = 4^2 + (-1)^2 + 2^2 = 21 \\
\]

Thus, the vector component of \( u \) along \( a \) is

\[
\text{proj}_a u = \frac{u \cdot a}{\|a\|^2} a = \frac{15}{21} (4, -1, 2) = \left(\frac{20}{7}, -\frac{5}{7}, \frac{10}{7}\right) \\
\]

and the vector component of \( u \) orthogonal to \( a \) is

\[
 u - \text{proj}_a u = (2, -1, 3) - \left(\frac{20}{7}, -\frac{5}{7}, \frac{10}{7}\right) = \left(-\frac{6}{7}, -\frac{2}{7}, \frac{11}{7}\right) \\
\]

Verify that the vector \( u - \text{proj}_a u \) and \( a \) are perpendicular by showing that their dot product is zero.
Example 7
Distance Between a Point and a Line (1/2)

Find a formula for the distance $D$ between point $P_0(x_0, y_0)$ and the line $ax + by + c = 0$.

Solution:
Let $Q(x_1, y_1)$ be any point on the line and position the vector $n = (a, b)$ so that its initial point is at $Q$.

By virtue of Example 5, the vector $n$ is perpendicular to the line (Fig 3.3.8). As indicated in the figure, the distance $D$ is equal to the length of the orthogonal projection of $QP_0$ on $n$; thus,

$$D = \left\| \text{proj}_n QP_0 \right\| = \frac{\left\| QP_0 \cdot n \right\|}{\|n\|}$$

But

$$QP_0 = (x_0 - x_1, y_0 - y_1), \quad QP_0 \cdot n = a(x_0 - x_1) + b(y_0 - y_1), \quad \|n\| = \sqrt{a^2 + b^2}$$
Example 7
Distance Between a Point and a Line (2/2)

Solution (count)
so that

\[ D = \frac{|ax_0 - x_1 + by_0 - y_1|}{\sqrt{a^2 + b^2}} \]  \hspace{1cm} (12)

Since the point \( Q(x_1, y_1) \) lies on the line, its coordinates satisfy the equation of the line, so

\[ ax_1 + by_1 + c = 0 \quad \text{or} \quad c = -ax_1 - by_1 \]

Substituting this expression in (12) yields the formula

\[ D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} \]  \hspace{1cm} (13)
Example 8
Using the Distance Formula

It follows from Formula (13) that the distance $D$ from the point $(1,-2)$ to the line $3x + 4y - 6 = 0$ is

$$D = \frac{|(3)(1) + 4(-2) - 6|}{\sqrt{3^2 + 4^2}} = \frac{|-11|}{\sqrt{25}} = \frac{11}{5}$$
4 Cross Product
Cross Product of Vectors

- Recall from Section 3 that the dot product of two vectors in 2-space or 3-space produces a **scalar**.
- We will now define a type of vector multiplication that produces a **vector** as the product, but which is applicable only in 3-space.
Definition

If \( \mathbf{u} = (u_1, u_2, u_3) \) and \( \mathbf{v} = (v_1, v_2, v_3) \) are vectors in 3-space, then the \textit{cross product} \( \mathbf{u} \times \mathbf{v} \) is the vector defined by

\[
\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)
\]

or in determinant notation

\[
\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \quad -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \quad \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}
\]

(1a)
Example 1

Calculating a Cross Product

Find $\mathbf{u} \times \mathbf{v}$, where $\mathbf{u} = (1, 2, -2)$ and $\mathbf{v} = (3, 0, 1)$.

Solution.

From either (1) or the mnemonic in the preceding remark, we have

$$
\mathbf{u} \times \mathbf{v} = 
\begin{vmatrix}
2 & -2 \\
0 & 1 \\
\end{vmatrix}
- 
\begin{vmatrix}
1 & -2 \\
3 & 1 \\
\end{vmatrix}
\begin{vmatrix}
1 & 2 \\
3 & 0 \\
\end{vmatrix}

= (2, -7, -6)
$$
Theorem 4.1
Relationships Involving Cross Product and Dot Product

If \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) are vectors in 3-space, then:

(a) \( \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0 \)  
(b) \( \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0 \)  
(c) \( \|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 \)  
(d) \( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \)  
(e) \( (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} \)
Example 2
\( \mathbf{u} \times \mathbf{v} \) Is Perpendicular to \( \mathbf{u} \) and to \( \mathbf{v} \)

Consider the vectors
\[
\mathbf{u} = (1, 2, -2) \quad \text{and} \quad \mathbf{v} = (3, 0, 1)
\]

In Example 1 we showed that
\[
\mathbf{u} \times \mathbf{v} = (2, -7, -6)
\]

Since
\[
\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = (1)(2) + (2)(-7) + (-2)(-6) = 0
\]

and
\[
\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = (3)(2) + (0)(-7) + (1)(-6) = 0
\]

\( \mathbf{u} \times \mathbf{v} \) is orthogonal to both \( \mathbf{u} \) and \( \mathbf{v} \) as guaranteed by Theorem 3.4.1.
Theorem 4.2
Properties of Cross Product

If \( u, v, \) and \( w \) are any vectors in 3-space and \( k \) is any scalar, then:

(a) \( u \times v = -(v \times u) \)
(b) \( u \times (v + w) = (u \times v) + (u \times w) \)
(c) \( (u + v) \times w = (u \times w) + (v \times w) \)
(d) \( k(u \times v) = (ku) \times v = u \times (kv) \)
(e) \( u \times 0 = 0 \times u = 0 \)
(f) \( u \times u = 0 \)
Example 3

Standard Unit Vectors

Consider the vectors

\[ \mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1) \]

These vectors each have length 1 and lie along the coordinate axes (Figure 3.4.1). They are called the **standard unit vectors** in 3-space. Every vector \( \mathbf{v} = (v_1, v_2, v_3) \) in 3-space is expressible in terms of \( \mathbf{i} \), \( \mathbf{j} \), and \( \mathbf{k} \) since we can write

\[ \mathbf{v} = (v_1, v_2, v_3) = v_1(1, 0, 0) + v_2(0, 1, 0) + v_3(0, 0, 1) = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} \]

For example,

\[ (2, -3, 4) = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k} \]

From (1b) we obtain

\[ \mathbf{i} \times \mathbf{j} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \\ \end{vmatrix} , \quad \begin{vmatrix} 1 & 0 \\ 0 & 0 \\ \end{vmatrix} , \quad \begin{vmatrix} 1 & 0 \\ 0 & 1 \\ \end{vmatrix} \right) = (0, 0, 1) = \mathbf{k} \]
Example 3
Standard Unit Vectors (count)

- Obtaining the following results:

\[
\begin{align*}
    i \times i &= 0 & j \times j &= 0 & k \times k &= 0 \\
    i \times j &= k & j \times k &= i & k \times i &= j \\
    j \times i &= -k & k \times j &= -i & i \times k &= -j
\end{align*}
\]

- Figure 3.4.2 is helpful for remembering these results. Referring to this diagram, the cross product of two consecutive vectors going clockwise is the next vector around, and the cross product of two consecutive vectors going counterclockwise is the negative of the next vector around.
A cross product can be represented symbolically in the form of a $3 \times 3$ determinant:

\[
\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}
\]

For example: if $\mathbf{u} = (1,2,-2)$ and $\mathbf{v} = (3,0,1)$, then

\[
\mathbf{u} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix} = 2i - 7j - 6k
\]
Determinant Form of Cross Product (2/2)

Warning: It is not true in general that
\[ u \times (v \times w) = (u \times v) \times w \]

For example
\[ i \times (j \times j) = i \times 0 = 0 \]
and
\[ (i \times j) \times j = k \times j = -i \]
so that
\[ i \times (j \times j) \neq (i \times j) \times j \]

We known from Theorem 3.4.1 that \( u \times v \) is orthogonal to both \( u \) and \( v \). If \( u \) and \( v \) are nonzero vectors, it can be shown that the direction of \( u \times v \) can be determined using the following right-hand rule.
Geometric Interpretation of Cross Product

- If \( \mathbf{u} \) and \( \mathbf{v} \) are vectors in 3-space, then the norm of \( \mathbf{u} \times \mathbf{v} \) has a useful geometric interpretation. Lagrange's identity, given in Theorem 3.4.1, states that

\[
\| \mathbf{u} \times \mathbf{v} \|^2 = \| \mathbf{u} \|^2 \| \mathbf{v} \|^2 - (\mathbf{u} \cdot \mathbf{v})^2
\]

\[
= \| \mathbf{u} \|^2 \| \mathbf{v} \|^2 - \| \mathbf{u} \|^2 \| \mathbf{v} \|^2 \cos^2 \theta
\]

\[
= \| \mathbf{u} \|^2 \| \mathbf{v} \|^2 (1 - \cos^2 \theta) = \| \mathbf{u} \|^2 \| \mathbf{v} \|^2 \sin^2 \theta
\]

since \( 0 \leq \theta \leq \pi \), it follows that \( \sin \theta \geq 0 \), so

\[
\| \mathbf{u} \times \mathbf{v} \| = \| \mathbf{u} \| \| \mathbf{v} \| \sin \theta
\]
Theorem 4.3
Area of a Parallelogram

If $u$ and $v$ are vectors in 3-space, then $\|u \times v\|$ is equal to the area of the parallelogram determined by $u$ and $v$. 
Example 4

Area of a Triangle

Find the area of the triangle determined by the points $P_1(2, 2, 0)$, $P_2(-1, 0, 2)$, and $P_3(0, 4, 3)$.

Solution.

The area $A$ of the triangle is $\frac{1}{2}$ the area of the parallelogram determined by the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ (Figure 3.4.5). Using the method discussed in Example 2 of Section 3.1, $\overrightarrow{P_1P_2} = (-3, -2, 2)$ and $\overrightarrow{P_1P_3} = (-2, 2, 3)$. It follows that

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = (-10, 5, -10)$$

and consequently

$$A = \frac{1}{2} \| \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} \| = \frac{1}{2}(15) = \frac{15}{2}$$
Definition

If \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) are vectors in 3-space, then

\[
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})
\]

is called the \textit{scalar triple product} of \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w}. \)

\[
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{u} \cdot \left( \begin{vmatrix} v_2 & v_3 & 1 \\ w_2 & w_3 & 1 \\ 1 & 1 & 1 \end{vmatrix} \right) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}
\]

(7)
Example 5
Calculating a Scalar Triple Product (1/2)

Calculate the scalar triple product $u \cdot (v \times w)$ of the vectors

$$u = 3i - 2j - 5k, \quad v = i + 4j - 4k, \quad w = 3j + 2k$$

**Solution.**

From (7)

$$u \cdot (v \times w) = \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 4 & -4 \\ 3 & 2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & -4 \\ 0 & 2 \end{vmatrix} + (-5) \begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix}$$

$$= 60 + 4 - 15 = 49$$
Example 5
Calculating a Scalar Triple Product (2/2)

- Note: The symbol \((u \cdot v) \times w\) make no sense since we cannot form the cross product of a scale and a vector.
- It follows from (7) that

\[
  u \cdot (v \times w) = w \cdot (u \times v) = v \cdot (w \times u)
\]

This relationship can be remembered by moving the vector \(u, v,\) and \(w\) clockwise around the vertices of the triangle in Figure 3.4.6.