Two Extremal Problems for Oriented Trellises

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1 Introduction

Consider the following type of ranked posets $P$. Each level $P_i$ of $P$ consists of $n$ vertices, which we denote by $a_{i1}, ..., a_{in}$. The partial order on $P$ is defined as follows: $a_{ij} \prec a_{ik}$ for $i > 1$ iff $a_{0j} \prec a_{1k}$. In other words, the Hasse diagram of $P$ consists of repetitions of the fragment of this diagram, restricted to the first bottom levels of $P$. It gives a reason to call such a poset $n$-periodic. Denote by $G = (V, E)$ the bipartite subgraph of the Hasse diagram of $P$, induced by the bipartition sets $P_0, P_1$.

We call a vertex $a_{0j} \in P_0$ reflexive if $a_{0j} \prec a_{ij}$ for some $i > 0$. The problems we consider concern the existence of reflexive vertices in $P_0$, depending on $|E|$. Such type of problems are useful for the theory of convolutional codes.

Denote by $\lambda(n)$ the minimal number, such that in any $n$-periodic poset with $|E| = \lambda(n)$ there exists a reflexive vertex $a \in P_0$. Similarly, let $\mu(n)$ denotes the minimal number, such that in any $n$-periodic poset with $|E| = \mu(n)$ each vertex $a \in P_0$ is reflexive.

Theorem 1 $\lambda(n) = \binom{n}{2} + 1$.

Theorem 2 $\mu(n) = n^2 - n + 1$.

2 Proofs of the theorems

Let us consider the graph $G$ as oriented graph, assuming that an edge $(a_{0i}, a_{1j})$ has orientation in the direction from $a_{0i}$ to $a_{1j}$. Now let us extend $G$ to $\tilde{G}$, introducing $n$ edges of the form $a_{1i} \rightarrow a_{0i}$, $i = 1, ..., n$. We call such edges inverse edges, and the original edges of $G$ the direct edges. Clearly, a vertex $a_{0i}$ is reflexive iff there exists an oriented cycle in $\tilde{G}$ passing through it.

For $a \in P_0$ and $b \in P_1$ denote

$$T(a) = \{c \in P_1 \mid (a, c) \in E\},$$

$$P(b) = \{c \in P_0 \mid (c, b) \in E\}.$$}

Further denote $a_{1i} = \pi(a_{0i})$ and $a_{0i} = \pi(a_{1i})$. For $A \subseteq P_0$ (or $A \subseteq P_1$) let $\pi(A) = \bigcup_{a \in A} \pi(a)$.

Lemma 1 If no oriented cycle pass through $a \in P_0$ in $\tilde{G}$, then $T(a) \cap \pi(P(\pi(a))) = \emptyset$.

Indeed, existence of a vertex $b \in T(a) \cap \pi(P(\pi(a)))$ implies existence of an oriented cycle in $\tilde{G}$ of length 4: $a \rightarrow b \rightarrow \pi(b) \rightarrow \pi(a) \rightarrow a$.

Proof of Theorem 1. By Lemma 1 we have to prove that if there exists an oriented cycle in $\tilde{G}$, then $|E| \geq \binom{n}{2} + 1$. We use induction on $n$. For $n = 2$ the Theorem is, obviously, true. Let $\tilde{G}$ be acyclic and estimate the number of its direct edges.
Let \( a \in P_0 \). Consider the graph \( \tilde{G}' \) obtained from \( \tilde{G} \) by removal all edges of the form \((a, b)\) and \((b, \pi(a))\). Then \( \tilde{G}' \) is also acyclic, and by induction the number of its direct edges is at most \( \binom{n-1}{2} \). Applying Lemma 1, the number of direct edges in \( \tilde{G} \) passing through \( a \) or \( \pi(a) \) (i.e. the removed edges) is at most \( n - 1 \). Thus the total number of direct edges in \( \tilde{G} \) is at most \( \binom{n-1}{2} + n - 1 = \binom{n}{2} \).

Finally, let us mention that for an acyclic graph the bound \(|E| \leq \binom{n}{2}\) is attainable, i.e. there exists an acyclic graph with exactly such much direct edges. To construct such a graph, connect the vertex \( a_i^0 \) with \( a_j^1 \) for \( i = 1, ..., n - 1, \ j = i + 1, ..., n \) (cf. Fig. 1a for \( n = 5 \), the inverse edges are shown in dashed lines). To prove the acyclicity of this graph, one has to notice that \( a_0^0 \) is connected by a direct edge with \( a_j^1 \) with \( j > i \), and so we cannot return to \( a_0^0 \) using the inverse edges.

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 a_0^0 \bullet \  a_1^1 \\
 a_2^0 \bullet \  a_2^1 \\
 a_3^0 \bullet \  a_3^1 \\
 a_4^0 \bullet \  a_4^1 \\
 a_5^0 \bullet \  a_5^1 \\
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 a_2^0 \bullet \  a_2^1 \\
 a_3^0 \bullet \  a_3^1 \\
 a_4^0 \bullet \  a_4^1 \\
 a_5^0 \bullet \  a_5^1 \\
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\( \text{Fig. 1} \)

**Proof of Theorem 2.** Let \( G \) be a graph with \(|E| \geq n^2 - n + 1\) and assume that there exists a vertex \( a \in G \) with no cycle passing through it. By Lemma 1 the number of direct edges in \( G \) incident with \( a \) or \( \pi(a) \) is at most \( n - 1 \). The number of edges incident with vertices \( V \setminus \{a, \pi(a)\} \) is at most \((n - 1)^2\) (as in the complete bipartite graph). Therefore, the total number of direct edges in \( G \) is at most \((n - 1)^2 + n - 1 = n^2 - n\). A contradiction.

On the other hand, in the following graph with \( n^2 - n \) direct edges, where the vertex \( a_i^0 \) is incident with \( a_j^1 \), \( i = 1, ..., n - 1, \ j = 1, ..., n \), the vertex \( a_n^0 \) is not reflexive (cf. Fig. 1b for \( n = 5 \)).

Therefore to guarantee the existence of an oriented cycle one has to have roughly half of the possible number of direct edges, and in order to each vertex \( a_i^0 \) to be reflexive, one has to have almost all possible edges in a graph. However, for some particular graphs, each vertex can be reflexive with much less number of direct edges.

Consider, for example, a graph, where each vertex \( a_i^0 \) is incident with exactly one edge, and each vertex \( a_i^1 \) is also incident just with one edge. Such a graph corresponds to a permutation, and the existence of an oriented cycle follows from representation of permutations as product of cycles.

As another example, consider a graph, where \(|T(A)| \geq |A|\) holds for any \( A \subseteq P_0 \). By Hall’s theorem on distinct representatives, there exists a perfect matching in such a graph. This matching corresponds to some permutation, which by above guarantees the existence of the cycles.