Specification of all Solutions of the Discrete Isoperimetric Problem that Have a Critical Cardinality

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Extended abstract

We present here a description of all solutions of the isoperimetric problem in Hamming space of some special cardinalities. The number of these cardinalities equals $2^{n-1}$.

Let $B^n$ denotes the vertex set of the $n$—dimensional unit cube with Hamming metric and $A \subseteq B^n$. Denote by $S_k^k(\alpha)$ the sphere of radius $k$ centered in $\alpha \in B^n$. We call a point $\alpha \in A$ the inner point of a set $A$ if $S_1^k(\alpha) \subseteq A$ and the boundary point of $A$ in the opposite case. Denote by $P(A)\Gamma(A)$ the collection of all inner (boundary) points of $A$.

Consider an isoperimetric problem to find for a fixed $m$, $1 \leq m \leq 2^n$, an $m$-element set $A \subseteq B^n$, such that $|\Gamma(A)| \leq |\Gamma(B)|$ for any $B \subseteq B^n$, $|B| = m$. We call such a set $A$ the optimal one. In [2] it is shown that the set $L(n,m)$ is a solution of the isoperimetric problem. $L(n,m)$ is defined as the initial segment of length $m$ of the following order of vertices of $B^n$. We say that a vertex $\alpha \in B^n$ precedes $\beta \in B^n$ iff $\|\alpha\| \leq \|\beta\|$, or if $\|\alpha\| = \|\beta\|$ then $\alpha$ is greater $\beta$ in the lexicographical order, where $\|\alpha\|$ is the coordinate sum of $\alpha$.

For $A \subseteq B^n$ we call a point $\alpha \subseteq A$ the free point of $A$ if $P(A) = P(A\setminus\alpha)$ and denote by $S(A)$ the collection of them. If $S(A) = \emptyset$, then a set $A$ is called critical.

Lemma 1 Let $A$ is an optimal noncritical set and $\alpha$ is it’s free point. Then the set $A\setminus\alpha$ is optimal either and $S(A\setminus\alpha) = S(A)\setminus\alpha$.

Corollary 1 $|S(A)| \leq |S(L(n,|A|))|.$

Corollary 2 $L(n,m)$ is a critical set then any optimal $m$-element subset of $B^n$ is critical too.

We call a number $m$ critical cardinality if $L(n,m)$ is a critical set.

Let $m$ and $n$ be fixed and $m^*$ be the greatest critical cardinality less or equal to $m$. It follows from Lemma 1 that if $A$ is an optimal critical $r$-element subset, $m^* \leq r \leq m$, then adding to it arbitrary $m - r$ points from $B^n \setminus A$ we get an optimal $m$-element set. Therefore the description of all optimal subset of $B^n$ is reduced to the description of all critical optimal sets only.

A specification of all optimal subsets meets a lot of difficulties. For example it turned out that in some cases the set $P(A)$ may be unconnected. Indeed, if there exists an integer
solution of equation $\Gamma(n, x) + \Gamma(n, m - x) = \Gamma(n, m)$ ($\Gamma(n, m) = |\Gamma(A)|$ for an optimal $m$-element set $A \subseteq B^n$), then the set $A = B \cup C$ for $B = L(n, m - x), C = B^n \setminus L(n, 2^n - x)$ is optimal. Moreover, since $B \cap C = \emptyset$ then at least for $m \leq 2^{n-1}$ there is a wide freedom for embedding of these parts into $B^n$. However, if we choose optimal sets with another structure as $B$ and $C$ then it is not clear whether it is possible to embed them together in $B^n$ with empty intersection. Similar situation occurs when $A$ may be divided to $t \geq 2$ nonintersecting pieces. A necessary condition of such division is the existence of integer solutions of equation $\Gamma(n, x_1) + \Gamma(n, x_2) + \cdots + \Gamma(n, x_t) = \Gamma(n, m)$ under condition $x_1 + \cdots + x_t = m$.

There are examples of $m$-element optimal sets which are either unconnected or connected for the same $m$, and at the last case their structure is not similar to the structure of $L(n, m)$ in the sense that they cannot be obtained from $L(n, m)$ by means of isometric transformations of $B^n$.

This paper is devoted to the specification of the family of optimal subsets of critical cardinalities.

Let us split the $n$-cube by the $i$-th coordinate $x_i$ into two $(n - 1)$-cubes and denote by $A^0(i)$ and $A^1(i)$ the parts of a set $A \subseteq B^n$ in these subcubes. We say that $A$ is $i$-normalized if $|A^0(i)| \geq |A^1(i)|$. A set which is $i$-normalized for $i = 1, \ldots, n$ is called simply normalized. Denote by $K(n, m)$ the collection of all $m$-element optimal subsets of critical cardinality $m$ and by $K(n, m)$ the collection of all normalized subsets in $K(n, m)$. It is clear that each subset $A \in K(n, m) \setminus K(n, m)$ may be transformed to some $B \in K(n, m)$ by "shifting" it coordinatewise modulo 2 on some binary vector $\gamma$. The $i$-th coordinate of $\gamma$ equals 0 iff $A$ is $i$-normalized and 1 otherwise.

**Lemma 2** $|K(n, m)| \leq 2^n \cdot |K(n, m)|$.

Let us assume that the number $m$ is represented in the form $m = \sum_{i=0}^{k} \binom{n}{i} + \delta$ with $0 \leq \delta < \binom{n}{k+1}$. We call a coordinate $x_i$ minimal coordinate of a set $A \subseteq B^n$ if $\|A^0(i)\| - |A^1(i)| \leq \|A^0(j)\| - |A^1(j)|$ for any $j = 1, \ldots, n$. The following lemma is proved in [1].

**Lemma 3** Let $A \in K(n, m)$ and $x_i$ be its minimal coordinate. Then $A^0(i)$ and $A^1(i)$ are optimal subsets (in $n - 1$ dimensions) of critical cardinalities and

$$|A^0(i)| = \sum_{i=0}^{k} \binom{n-1}{i} + \left( \delta \oplus \binom{n-1}{k} \right), \quad |A^1(i)| = \sum_{i=0}^{k} \binom{n-1}{i} - \left( \binom{n-1}{k} \oplus \delta \right),$$

where $a \oplus b$ equals $a - b$ if $a \geq b$ and 0 otherwise.

Our approach for the specification of all subsets of $K(n, m)$ is in the following. We introduce the concept of a division process of a set $A \in K(n, m)$. It consists of some steps and defines by induction on the number of step. On the first step we divide the set $A$ by it’s arbitrary minimal coordinate $x_{i_1}$. It follows from Lemma 3 that the set $A^0(i_1)$ (when $\delta < \binom{n-1}{k}$), or the set $A^1(i_1)$ (when $\delta \geq \binom{n-1}{k}$) is a sphere. The structure of the other set ($A^1(i_1)$ resp. $A^0(i_1)$) is unclear yet. We call this set unknown. Assume that after $l - 1$
Let \( l \geq 2 \) steps we have the only unknown set in some \((n - l + 1)\)-subcube and all the other parts of \( A \) in such subcubes are spheres. Then on the \( l \)-th step of our process we split \( B^n \) into \((n - l)\)-subcubes by arbitrary minimal coordinate \( x_i \) of the unknown set. From Lemma 3 it follows again, that we obtain no more than one unknown set in such a way and the other parts of \( A \) in the corresponding \((n - l)\)-subcubes are spheres. We call them \( l \)-spheres. The process terminates on the \( t \)-th step if the unknown set is a sphere.

**Lemma 4** Let \( A \in K(n, m) \). Then

(i) There exist numbers \( l_1, m_1, l_2, m_2, \ldots, l_r, m_r \) \((0 < l_1 \leq m_1 < l_2 \leq m_2, \ldots, < l_r \leq m_r)\), such that \( \delta \) may be uniquely represented in the following canonical form

\[
\delta = \left( \frac{n - l_1}{k - l_1 + 1} \right) + \left( \frac{n - l_2}{k - l_1 + 1} \right) + \cdots + \left( \frac{n - m_1}{k - l_1 + 1} \right) + \\
\left( \frac{n - l_2}{k - l_1 - l_2 + m_1 + 2} \right) + \cdots + \left( \frac{n - m_i}{k - l_1 - l_2 + m_1 + 2} \right) + \\
\left( \frac{n - l_r}{k - \sum_{i=1}^{r} l_i + \sum_{i=1}^{r-1} m_i + r} \right) + \cdots + \left( \frac{n - m_r}{k - \sum_{i=1}^{r} l_i + \sum_{i=1}^{r-1} m_i + r} \right);
\]

(ii) The number of steps in the splitting process equals \( m_r \).

We call a sequence \( I \) of coordinates \( x_{i_1}, \ldots, x_{i_m} \) admissible (here \( m = m_r \)) for a set \( A \in K(n, m) \) if there exists a splitting process, such that on the \( j \)-th step of it we split \( B^n \) by the coordinate \( x_j \), for \( j = 1, \ldots, m_r \). Denote the collection of admissible sequences for \( A \) by \( I(A) \) and for \( I \in I(A) \) let \( U_0(A, I) \) be the unknown set obtained after \( t \) steps of the splitting process. Consider the class \( L(n, m) \) of subset \( A \in K(n, m) \), such that for all \( t = 1, \ldots, m_r \) the set \( U_t(A, I) \) is normalized (in \((n - t)\) dimensions) for some \( I \in I(A) \).

Denote by \( \varphi(A) \) the set obtained from \( A \) by a permutation \( \varphi \) of coordinates of \( B^n \).

**Lemma 5**

(i) If \( A \in L(n, m) \), then \( A = \varphi(L(n, m)) \) for some permutation \( \varphi \);

(ii) \( |L(n, m)| = \frac{m!}{(l_1-1)!(m_1-l_1+1)! \cdots (l_r-m_r+1)!(m_r-l_r+1)!(n-m_r)!} \).

It turned out that that all the \( m_r \)-spheres (maybe except of \( U_{m_r}(A, I) \)) of \( A \in K(n, m) \) are centered at the origins of the corresponding \((n - m_r)\)-subcubes. The center \( \alpha \) of the sphere \( U_{m_r}(A, I) \) is either at the origin of the corresponding subcube (point \( \beta \)) or at some point \( \gamma \) of norm 1 (in this subcube). Notice, that if \( \alpha = \beta \) then \( A \in L(n, m) \). Denote by \( L(A) \in L(n, m) \) a set for which there exists a splitting process with the same sequence, i.e. \( I(A) \cap I(L(A)) \neq \emptyset \). Then \( A \) may be obtained from \( L(A) \) by some transposition of the \( m_r \)-spheres and maybe by replacing the center \( \alpha \) of the sphere \( U_{m_r}(A, I) \) to a point \( \gamma \). Therefore, the specification of all the subsets from \( K(n, m) \setminus L(n, m) \) may be reduced to specification of such transformations.

Let \( \gamma \in B^n \). Denote by \( C(\gamma, 1) \) the \((n - \|\gamma\|)\)-subcube of \( B^n \) including the points \( \gamma \) and \( 1 = (1, \ldots, 1) \). We call two \( m_r \)-spheres \( S_1 \) and \( S_2 \) \( i \)-neighboring if the origins of the corresponding \((n - m_r)\)-subcubes differ in the \( i \)-th entry only. Therefore for fixed \( i \) the set of \( m_r \)-spheres is divided into pairs of \( i \)-neighboring spheres. Define the transformation
\[ R(\alpha, x_i)A \] of a set \( A \in \mathcal{K}(n, m) \), which is in the following (here \( I = \{ x_{i_1}, \ldots, x_{i_m} \in I(A) \) and \( m = m_r \)):

1. If \( ||\alpha|| < m_r \) and \( x_i \in I \), then consider all the pairs of \( i \)-neighboring spheres, which are included into subcube \( C(\alpha, 1) \). Let \((S_1, S_2)\) be such a pair and these spheres are in \((n - m_r)\)-subcubes \( D_1 \) and \( D_2 \) respectively. Then replace \( S_1 \) to \( S_2 \) in subcube \( D_1 \) and replace \( S_2 \) to \( S_1 \) in \( D_2 \). Proceed analogously for all other pairs of \( i \)-neighboring spheres;

2. If \( ||\alpha|| = m_r \) and \( x_i \not\in I \), then invert the \( i \)-th entry of all points of the set \( U_{m_r}(A, I) \);

3. Otherwise the transformation is undefined.

We say that points \( \alpha_1, \ldots, \alpha_q \) and coordinates \( x_1, \ldots, x_q \) of \( B^n \) satisfy the condition \( W \) if

1. \( 1 \leq ||\alpha_1|| < ||\alpha_2|| < \cdots < ||\alpha_q|| \leq m_r \);
2. \( C(\alpha_1, 1) \supseteq C(\alpha_2, 1) \supseteq \cdots \supseteq C(\alpha_q, 1) \);
3. \( \{ x_1, \ldots, x_q \} \subseteq I \) if \( ||\alpha_q|| < m_r \) and \( \{ x_1, \ldots, x_{q-1} \} \subseteq I, x_q \not\in I \) if \( ||\alpha_q|| = m_r \).

**Theorem 1** For any \( A \in \mathcal{K}(n, m) \) there exist points \( \gamma_1, \ldots, \gamma_p \) and coordinates \( y_1, \ldots, y_p \), satisfying the condition \( W \), such that

\[ A = R(\gamma_p, y_p)R(\gamma_{p-1}, y_{p-1})\ldots R(\gamma_1, y_1)B \]

for some \( B \in \mathcal{L}(n, m) \).

Now we are going to show a way how to determine the points \( \gamma_1, \ldots, \gamma_p \) and coordinates \( y_1, \ldots, y_p \) for a set \( A \in \mathcal{K}(n, m) \) \( \mathcal{L}(n, m) \). Notice that if a set \( U_{m_r}(A, I) \) is not \( i \)-normalized then such coordinate \( x_i \) is unique and is the minimal coordinate for it. Denote by \( N(A, I) \) the collection of all indices \( a_1, \ldots, a_t \), such that \( U_{a_r}(A, I) \) is not normalized, and let \( N(A) = \bigcap_{i \in I} N(A, I) \).

**Theorem 2** For any set \( A \in \mathcal{K}(n, m) \) there exists a sequence \( I_A = \{ x_{i_1}, \ldots, x_{i_m} \} \) (with \( m = m_r \)), such that

1. \( N(A, I_A) = N(A) = \{ a_1, \ldots, a_t \} \);
2. the set \( U_{a_j}(A, I_A), 1 \leq j \leq t \), is not \( a_j \)-normalized, and if \( a_j < m_r \), then \( a_j = x_{i_{j+1}} \) holds.

Consider now a sequence \( I_A = \{ x_{i_1}, \ldots, x_{i_m} \} \) (with \( m = m_r \)), and the set \( N(A) = \{ j_1, \ldots, j_r \} \). Assume \( j_1 < j_2 < \cdots < j_r \). Consider vector \( \kappa = (\kappa_1, \ldots, \kappa_n) \in B^n \), such that \( \kappa_l = 0 \) iff \( l \not\in \{ i_1, \ldots, i_m \} \) \( m = m_r \), or if \( l = i_s \) for some \( s, 1 \leq s \leq m_r \), and in the canonical representation of \( \delta \) there is a binomial coefficient of the form for some \( c \). Then the \( i \)-th entry of \( \gamma_q \), \( q = 1, \ldots, t \), equals \( \kappa_i \) if \( i < j_q \) and 0 otherwise.

**References**
